426(4): Hamilton Jacobi Equation on the Classical Level

In order to apply the HJE to a theory we first consider the gravitational Hamiltonian:

$$H = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} - \frac{mMG}{r} \quad -(1)$$

In the Hamilton Jacobi theory:

$$p_r = \partial S_r / \partial r \quad -(2)$$

and

$$L = \partial S_\phi / \partial \phi \quad -(3)$$

The total action is:

$$S = W(r) + W(\phi) - Et \quad -(4)$$

where E is the total energy and where $W(r)$ and $W(\phi)$ are Hamilton's characteristic functions for the system.

The Hamilton Jacobi equation is therefore: $-(5)$

$$\frac{1}{2m}\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S_\phi}{\partial \phi}\right)^2 + \frac{\partial S_0}{\partial t} - \frac{mMG}{r} = 0$$

in which the actions S_0, S_r and S_ϕ are separable.

Now write

$$\frac{\partial S_0}{\partial t} = -E \quad -(6)$$

and

$$\frac{\partial S_\phi}{\partial \phi} = L \quad -(7)$$

where E is the total energy and L the angular momentum.

) The total energy is the Hamiltonian:

$$H = E \quad -(8)$$

and is a constant of motion:

$$\frac{dH}{dt} = 0 \quad -(9)$$

Eq. (9) is the first Evans Eckardt equation. The angular momentum (7) is also a constant of motion, so:

$$\frac{dL}{dt} = 0 \quad -(10)$$

This is the second Evans Eckardt equation. Eq. (5) therefore becomes:

$$H = \frac{1}{2m}\left(\frac{\partial S_r}{\partial r}\right)^2 + U(r) + \frac{L^2}{2mr^2} = E \quad -(11)$$

If the interaction potential is:

$$U(r) = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(12)$$

Eq. (11) can be transformed into:

$$\frac{d^2 \rho}{dr^2} + \frac{2m}{\hbar^2} V_{eff}\, \rho = -\frac{2mE}{\hbar^2}\rho \quad -(13)$$

using the quantization conditions:

$$\left(\frac{\partial S_r}{\partial r}\right)^2 \psi = -\hbar^2 \frac{\partial^2 \psi}{\partial r^2} \quad -(14)$$

and

$$L^2 = \ell(\ell+1)\hbar^2 \quad -(15)$$

3) So the Hamilton Jacobi equation (11) becomes the Schroedinger equation:

$$H\psi = -\frac{\hbar^2}{2m}\nabla^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi \quad -(16)$$

where

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi) \quad -(17)$$

and

$$P(r) = rR(r) \quad -(18)$$

With these definitions the Hamilton Jacobi equation gives the energy levels of the H atom.

The original Hamilton equations:

$$\dot{p}_r = -\frac{\partial H}{\partial r} \quad -(19)$$

and

$$\dot{L} = -\frac{\partial H}{\partial \phi} \quad -(20)$$

have been transformed to:

$$\dot{P}_r = -\frac{\partial H}{\partial Q_r} = 0 \quad -(21)$$

and

$$\dot{L} = -\frac{\partial H}{\partial \phi} = 0 \quad -(22)$$

using

$$H + \frac{\partial S}{\partial t} = 0 \quad -(23)$$

and

$$p = \frac{\partial S}{\partial q} \quad -(24)$$

7) The constants of motion are given by eqs. (6) and (7), which lead to the Evans Eckardt equations:

$$\frac{dH}{dt} = 0 \quad -(25)$$

and

$$\frac{dL}{dt} = 0 \quad -(26)$$

So:

$$\frac{d}{dt}\left(\frac{\partial S_0}{\partial t}\right) = 0 \quad -(27)$$

and

$$\frac{d}{dt}\left(\frac{\partial S_\phi}{\partial \phi}\right) = 0 \quad -(28)$$

but

$$\frac{d}{dt}\left(\frac{\partial S_r}{\partial r}\right) \neq 0 \quad -(29)$$

The action S_r can be found by integrating eq. (11) using a computer, or by recognizing that S_r is related to the associated Legendre functions, the radial wave functions of the H atom. The action S_ϕ is found by integrating:

$$\frac{\partial S_\phi}{\partial \phi} = L = mr^2\dot{\phi} \quad -(30)$$

a computer. The action S_0 is found by integrating:

$$\frac{\partial S_0}{\partial t} = -E \quad -(31)$$

where for example E are the energy levels of the H atom.

5) The quantum of action is $\hbar$, the reduced Planck constant, so action has the same units as angular momentum. The action is therefore fundamental to all of classical physics, the Hamilton Principle of Least Action being:

$$\delta \int_{t_1}^{t_2} \mathcal{L} \, dt = 0 \quad - (32)$$

where in general:

$$S = \int \mathcal{L} \, dt \quad - (33)$$

In order to extend these well known principles to special relativity and ECE theory, eq. (1) must be generalized, and the constants of motion and actions defined. Finally, we look for new information on ECE theory.
