326(5): Free Particle Quantization

In the case of the free particle:

$$H = \gamma mc^2 = E \quad -(1)$$

and

$$\mathcal{L} = -\frac{mc^2}{\gamma} \quad -(2)$$

Therefore:

$$H_1 = E - mc^2 = (\gamma - 1)mc^2 \quad -(3)$$

The total energy is defined by:

$$E^2 = p^2c^2 + m^2c^4 \quad -(4)$$

so

$$E^2 - m^2c^4 = p^2c^2 = H^2 - m^2c^4 \quad -(5)$$

i.e.

$$H_1 = T = E - mc^2 = (\gamma - 1)mc^2$$

$$= \frac{p^2c^2}{E + mc^2} \quad -(6)$$

The relativistic kinetic energy is:

$$H_1 = T = (\gamma - 1)mc^2 = \frac{p^2c^2}{(\gamma+1)mc^2} \quad -(7)$$

$$= \frac{p^2}{m(\gamma+1)}$$

Therefore

$$\frac{p^2}{m} = (\gamma^2 - 1)mc^2 \quad -(8)$$

$$= (\gamma^2 - 1)m^2c^2$$

2)

$$= (\gamma+1)(\gamma-1)mc^2 \quad -(9)$$

In the limit:

$$v << c, \quad -(10)$$

we have $$(\gamma-1)mc^2 \rightarrow \frac{1}{2}mv^2 \quad -(11)$$

and $$(\gamma+1) \rightarrow 2 \quad -(12)$$

so $$\frac{p^2}{2m} \rightarrow \frac{1}{2}mv^2 \quad -(13)$$

Q.E.D.

In the case of a free particle the total energy is

$$E = T \quad -(14)$$

so $$\boxed{\frac{p^2}{2m} = \frac{1}{2}(1+\gamma)E} \quad -(15)$$

In the non relativistic limit:

$$\frac{p^2}{2m} = T = E \quad -(16)$$

On quantization, eq. (16) gives the free particle Schroedinger equation:

$$-\frac{\nabla^2\hbar^2\phi}{2m} = E\phi \quad -(17)$$

The solution of eq. (17) for:

$$\frac{\partial^2 \phi}{\partial Z^2} = -\left(\frac{2mE}{\hbar^2}\right)\phi \quad -(18)$$

are

$$\phi = A\exp(i\kappa Z) + B\exp(-i\kappa Z) \quad -(19)$$

where

$$\kappa = \left(\frac{2mE}{\hbar^2}\right)^{1/2} \quad -(20)$$

So

$$E = \frac{\kappa^2\hbar^2}{2m} = \frac{p^2}{2m} \quad -(21)$$

and

$$\boxed{p = \hbar\kappa} \quad -(22)$$

This is the de Broglie wave particle dualism in the classical limit $v \ll c$.

In the relativistic case:

$$E^2 = \hbar^2\kappa^2 + m^2c^4 \quad -(23)$$

So

$$E - mc^2 = \frac{\hbar\kappa}{E + mc^2} \quad -(24)$$

So eq. (22) is also true in the relativistic free particle.

Eq. (7) is:

$$p^2 = (\gamma+1)mE \quad -(25)$$

4) In this equation:

$$p = \gamma m v = \gamma p_0 \quad -(26)$$

is the relativistic momentum, and:

$$p_0 = m v \quad -(27)$$

is the classical momentum

So:

$$\gamma^2 p_0^2 = (1+\gamma) m E \quad -(28)$$

and

$$\boxed{\frac{p_0^2}{2m} = \frac{1}{2}\left(\frac{1+\gamma}{\gamma^2}\right) E} \quad -(29)$$

In the limit

$$\gamma \rightarrow 1 \quad -(30)$$

this goes to:

$$\frac{p_0^2}{2m} = E = T \quad -(31)$$

In eq. (29):

$$\gamma = \left(1 - \frac{p_0^2}{m^2 c^2}\right)^{-1/2} \quad -(32)$$

and

$$p = \gamma p_0 = \hbar \kappa \quad -(33)$$

Using computer algebra, E can be solved in terms of κ from eqs. (29) and (33).

5) <u>Cross Check</u>

Eq. (7) is

$$\gamma^2 p_0^2 = (1+\gamma) m E = p^2 \quad -(34)$$

so

$$p^2 = (\gamma^2 - 1) m^2 c^2 \quad -(35)$$

and

$$p^2 c^2 = (\gamma^2 - 1) m^2 c^4 \quad -(36)$$

However:

$$E^2 = \gamma^2 m^2 c^4 = p^2 c^2 + m^2 c^4 \quad -(37)$$

so

$$\gamma^2 = \frac{1}{m^2 c^4} (p^2 c^2 + m^2 c^4)$$

$$= \frac{p^2}{m^2 c^2} + 1$$

$$= \gamma^2 \frac{p_0^2}{m^2 c^2} + 1 \quad -(38)$$

$$= \gamma^2 \frac{v^2}{c^2} + 1$$

So

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(39)$$

where

$$v = \frac{p_0}{m} \quad -(40)$$

QED