290(5): Integration of Higher Order Infinitesimals

This subject is discussed in chapter 16 of "Vector Analysis Problem Solver" in terms of volume integrals. The volume infinitesimal is:

$$dV = dx\,dy\,dz \quad -(1)$$

and if

$$x = y = z \quad -(2)$$

then

$$dV = (dx)^3 \quad -(3)$$

For example:

$$\int_0^1 \int_0^1 \int_0^1 x^2 y \, dx\,dy\,dz$$

$$= \int_0^1 \int_0^1 x^2 y \, dx\,dy$$

$$= \frac{1}{2} \int_0^1 x^2 \, dx = \frac{1}{6} \quad -(4)$$

This is example 16.3.

Eq. (4) may be written as:

$$\int_0^1 \int_0^1 \int_0^1 x^2 x \, 1 \, dx\,dx\,dx \quad -(5)$$

$$= \frac{1}{6}$$

The indefinite integrals in eq. (4) is:

2) $$\iiint x^2 y \, dx\, dy\, dz = \int x^2 dx \int y\, dy \int dz$$
$$= \frac{x^3 y^2 z}{6} \quad -(6)$$

Eq. (4) may So written as:
$$\int_{x=0}^{1} x^2 dx \int_{y=0}^{1} y\, dy \int_{z=0}^{1} dz = \frac{1}{6} \quad -(7)$$

This is the same as:
$$\int_{x=0}^{1} x^2 dx \int_{x=0}^{1} x\, dx \int_{x=0}^{1} dx = \frac{1}{6} \quad -(8)$$
$$= \int_0^1\int_0^1\int_0^1 x^2 x 1 (dx)^3$$

From note 290(4) the correct intensity from the Planck distribution is; for all frequencies (blackbody):
$$I = \frac{1}{\pi^2 c^2}\left[\int_0^\infty \omega^2 f(\omega)\, d\omega + \frac{2}{3}\int_0^\infty\int_0^\infty \omega f(\omega)(d\omega)^2 + \frac{1}{3}\int_0^\infty\int_0^\infty\int_0^\infty f(\omega)(d\omega)^3\right] \quad -(9)$$

where
$$f(\omega) = \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1} \quad -(10)$$

Second Order Integral

This is:

$$\frac{2}{3}\int_0^\infty \int_0^\infty \omega f(\omega)(d\omega)^2 \quad \text{—(11)}$$

$$= \frac{2}{3}\int_0^\infty \omega^{1/2} f^{1/2}(\omega)\, d\omega \int_0^\infty \omega^{1/2} f^{1/2}(\omega)\, d\omega$$

Third Order Integral

This is:

$$\frac{1}{3}\int_0^\infty \int_0^\infty \int_0^\infty f(\omega)(d\omega)^3 \quad \text{—(12)}$$

$$= \frac{1}{3}\int_0^\infty f^{1/3}(\omega)\, d\omega \int_0^\infty f^{1/3}(\omega)\, d\omega \int_0^\infty f^{1/3}(\omega)\, d\omega$$

High Frequency or Low Temperature Limit

As in eq. (9) of note 290(5) the intensity is:

$$I = \frac{kT}{\pi^2 c^2}\left[\int \omega^2 d\omega + \frac{2}{3}\iint \omega\, d\omega\, d\omega + \frac{1}{3}\iiint d\omega\, d\omega\, d\omega\right] \quad \text{—(13)}$$

The second order integral is:

4)

$$\frac{2}{3}\iint \omega \, d\omega \, d\omega = \frac{2}{3}\int \omega^{1/2} d\omega \int \omega^{1/2} d\omega \quad -(14)$$

and the third order integral is:

$$\frac{1}{3}\int d\omega \int d\omega \int d\omega = \frac{1}{3}\iiint (d\omega)^3 \quad -(15)$$

therefore the current intensity from eq. (9) is:

$$I = \frac{1}{\pi^2 c^2}\left[\int_0^\infty \omega^2 f(\omega)\, d\omega + \frac{2}{3}\left(\int_0^\infty \omega^{1/2} f^{1/2}(\omega)\, d\omega\right)^2 + \frac{1}{3}\left(\int_0^\infty f^{1/3}(\omega)\, d\omega\right)^3\right] \quad -(16)$$

where:

$$f(\omega) = \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1} \quad -(17)$$
