260(2): Beltrami Structures and Magnetic Monopoles

The magnetic flux density in general is defined by:

$$\underline{\nabla} \times \underline{B}^a = \kappa \underline{B}^a \quad -(1)$$

where

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \quad -(2)$$

Assuming a zero magnetic monopole implies:

$$\underline{\omega}^a{}_b \cdot \underline{B}^b = \underline{A}^b \cdot \underline{R}^a{}_b \, (\text{spin}), \quad -(3)$$

and

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{A}^b = 0 \quad -(4)$$

from the Cartan identity. Eq. (4) implies:

$$\underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{A}^b = \underline{A}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b \quad -(5)$$

and:

$$\underline{\nabla} \times (\underline{\omega}^a{}_b \times \underline{A}^b) = \kappa \, \underline{\omega}^a{}_b \times \underline{A}^b \quad -(6)$$

From eq. (2):

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b) = \kappa (\underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b) \quad -(7)$$

From eqs. (6) and (7):

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}^a) = \kappa \, \underline{\nabla} \times \underline{A}^a \quad -(8)$$

So

$$\underline{\nabla} \times \underline{A}^a = \kappa \underline{A}^a \quad -(9)$$

From eqs. (5) and (9):

$$\underline{\nabla} \times \underline{\omega}^a{}_b = \kappa \underline{\omega}^a{}_b \quad -(10)$$

The spin curvature is defined by:

$$\underline{R}^a{}_b(\text{spin}) = \underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \quad -(11)$$

so

$$\underline{\nabla} \cdot \underline{R}^a{}_b(\text{spin}) = \underline{\nabla} \cdot \left(\underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \right) \quad -(12)$$

$$= - \underline{\nabla} \cdot \left(\underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \right)$$

$$= \underline{\nabla} \times \underline{\omega}^c{}_b \cdot \underline{\omega}^a{}_c - \underline{\omega}^c{}_b \cdot \underline{\nabla} \times \underline{\omega}^a{}_c$$

$$= \kappa \left(\underline{\omega}^c{}_b \cdot \underline{\omega}^a{}_c - \underline{\omega}^c{}_b \cdot \underline{\omega}^a{}_c \right)$$

$$= 0 \quad -(13)$$

It follows that:

$$\underline{\nabla} \times \underline{R}^a{}_b(\text{spin}) = \kappa \underline{R}^a{}_b(\text{spin}) \quad -(14)$$

The absence of a magnetic monopole implies that $\underline{B}^a$, $\underline{A}^a$, $\underline{\omega}^a{}_b$ and $\underline{R}^a{}_b(\text{spin})$ are all Beltrami structures in general.

3) The magnetic monopole is defined by:

$$\rho^a_{mag} = \epsilon_0 c \left(\underline{\omega}^a{}_b \cdot \underline{B}^b - \underline{A}^b \cdot \underline{R}^a{}_b (\text{spin}) \right) \quad -(15)$$

so the magnetic monopole of an electron or proton or neutron has an internal structure characterized by

$$\rho^a_{mag} = 0 \quad -(16)$$

From eq. (3):

$$\underline{\omega}^a{}_b \cdot \left(\underline{\nabla} \times \underline{A}^b - \underline{\omega}^b{}_c \times \underline{A}^c \right) = \frac{1}{\kappa} \underline{\nabla} \times \underline{A}^b \cdot \underline{R}^a{}_b (\text{sp}) \quad -(17)$$

so

$$\underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{A}^b = \frac{1}{\kappa} \underline{\nabla} \times \underline{A}^b \cdot \underline{R}^a{}_b (\text{sp}) + \underline{\omega}^a{}_b \cdot \underline{\omega}^b{}_c \times \underline{A}^c \quad -(18)$$

$$= \frac{1}{\kappa} \underline{\nabla} \times \underline{A}^b \cdot \underline{R}^a{}_b \Big|^{(\text{sp})} + \underline{A}^c \cdot \underline{\omega}^a{}_b \times \underline{\omega}^b{}_c$$

$$= \frac{1}{\kappa} \underline{\nabla} \times \underline{A}^b \cdot \left(\underline{R}^a{}_b \Big|^{(\text{sp})} + \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \right)$$

so

$$\underline{R}^a{}_b \Big|^{(\text{sp})} = \kappa\, \underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b$$

$$= \underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \quad -(19)$$

This is a self consistent result and the correct definition of $\underline{R}^a{}_b(\text{spin})$. Therefore the absence of a magnetic monopole is equivalent to:

$$\underline{\nabla} \times \underline{R}^a{}_b(\text{spin}) = \kappa \, \underline{R}^a{}_b(\text{spin}) \quad -(20)$$

$$\underline{\nabla} \cdot \underline{R}^a{}_b(\text{spin}) = 0 \quad -(21)$$

so

$$\boxed{\left(\nabla^2 + \kappa^2\right) \underline{R}^a{}_b(\text{spin}) = \underline{0}} \quad -(22)$$

This is a Helmholtz equation in spin curvature. It gives structure to an electron, proton or neutron. The spin curvature inside these elementary particles is a very rich one, with well known solutions in terms of Bessel functions.