238(7): Definition of the Angular Momentum

The angular momentum is defined as:

$$L = mr^2 \frac{d\theta}{d\tau} = \gamma m r^2 \frac{d\theta}{dt} \quad -(1)$$

in which

$$\omega = \frac{d\theta}{dt} = \frac{d\theta}{dr}\frac{dr}{dt}. \quad -(2)$$

The quantity $d\theta / dr$ is observable from the planar orbit.

For example, for the ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(3)$$

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha}\sin\theta, \quad \frac{d\theta}{dr} = \frac{\alpha}{\epsilon r^2 \sin\theta}. \quad -(4)$$

The Lorentz factor is:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(5)$$

where

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$

$$= \left(\frac{dr}{dt}\right)^2\left(1 + \left(\frac{d\theta}{dr}\right)^2\right). \quad -(6)$$

So

$$\frac{dr}{dt} = \frac{v}{\left(1 + \left(\frac{d\theta}{dr}\right)^2\right)^{1/2}} \quad -(7)$$

2)

Therefore from eqs. (2) and (7):

$$\omega = \frac{d\theta}{dr}\left(1+\left(\frac{d\theta}{dr}\right)^2\right)^{-1/2} \quad -(8)$$

and

$$\boxed{L = \gamma m r^2 \frac{d\theta}{dr}\left(1+\left(\frac{d\theta}{dr}\right)^2\right)^{-1/2}} \quad -(9)$$

For the elliptical orbit:

$$r^2 = \frac{\alpha^2}{(1+\epsilon\cos\theta)^2} \quad -(10)$$

and

$$\frac{d\theta}{dr} = \frac{\alpha}{\epsilon r^2 \sin\theta} = \frac{(1+\epsilon\cos\theta)^2}{\epsilon\alpha\sin\theta} \quad -(11)$$

In eq. (1), the quantity:

$$L_0 = m r^2 \frac{d\theta}{dt} \quad -(12)$$

can be coded in as a constant. So:

$$L = \gamma L_0 \quad -(13)$$

where L_0 is a constant, the non-relativistic total angular momentum. Therefore:

$$L = \left(1-\frac{v^2}{c^2}\right)^{-1/2} L_0 \quad -(14)$$

3) Frm eq. (9) of note 238(4):

$$v^2 = \left(\frac{L_0}{m}\right)^2 \left(\frac{1}{r^2} + \left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2\right) \quad -(15)$$

So:

$$\boxed{L = \left(1 - \left(\frac{L_0}{mc}\right)^2 \left(\frac{1}{r^2} + \left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2\right)\right)^{-1/2} L_0} \quad -(16)$$

$$\gamma = \left(1 - \left(\frac{L_0}{mc}\right)^2 \left(\frac{1}{r^2} + \left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2\right)\right)^{-1/2} \quad -(17)$$

and:

$$\underline{F} = -\frac{L^2}{mr^3}\left(\gamma^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r + \frac{L^4}{m^3 r^3 c^2}\frac{d}{d\theta}\left(\frac{1}{r}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\underline{e}_\theta \quad -(18)$$

Code in L_0, m and c as constants.