238(5): Some Notes on the Derivation of the Force Equation

Consider the acceleration of any planar orbit in the non-relativistic limit:

$$\underline{a} = \frac{d^2 r}{dt^2}\,\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(1)$$

In this equation:

$$f = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{L}{mr^2}\frac{dr}{d\theta} \quad -(2)$$

and

$$\frac{df}{dt} = \frac{df}{dr}\frac{dr}{dt} = f\frac{df}{dr} \quad -(3)$$

So:

$$\frac{d^2 r}{dt^2} = \left(\frac{L}{mr}\right)^2 \left(\frac{dr}{d\theta}\right) \frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right). \quad -(4)$$

Note that

$$\frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) = \frac{d\theta}{dr}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) \quad -(5)$$

so

$$\frac{d^2 r}{dt^2} = \left(\frac{L}{mr}\right)^2 \frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right). \quad -(6)$$

Next note that:

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\right)\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} \quad -(7)$$

so

$$\frac{d^2 r}{dt^2} = -\left(\frac{L}{mr}\right)^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right). \quad -(8)$$

In eq. (1):

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\omega^2 r\,\underline{e}_r = -\frac{L^2}{m r^3}\underline{e}_r \quad -(9)$$

so:

$$\boxed{\underline{a} = -\left(\frac{L}{mr}\right)^2 \left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r} \quad -(10)$$