238(4): The θ Dependence of v of the Lorentz Factor

The Lorentz factor is:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(1)$$

where

$$v^2 = \frac{d\underline{r} \cdot d\underline{r}}{dt^2} \quad -(2)$$

This result is derived from:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(3)$$
$$= c^2 dt^2 - dr^2 - r^2 d\theta^2$$

and

$$d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\theta^2 \quad -(4)$$
$$= v^2 dt^2$$

So

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 \quad -(5)$$

and

$$c^2 d\tau^2 = (c^2 - v^2)\, dt^2, \quad -(6)$$

$$\frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \gamma. \quad -(7)$$

The velocity is:

$$\underline{v} = \frac{dr}{dt} \underline{e}_r + \omega r \underline{e}_\theta \quad -(8)$$
$$= \frac{dr}{dt} \underline{e}_r + \underline{\omega} \times \underline{r}.$$

2)

i.e

$$\underline{v} = \left(\frac{L}{m}\right)\left(\frac{1}{r}\underline{e}_\theta - \frac{d}{d\theta}\left(\frac{1}{r}\right)\underline{e}_r\right) \quad -(9)$$

For an elliptical orbit:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1+\epsilon\cos\theta\right) \quad -(10)$$

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\sin\theta \quad -(11)$$

So:

$$\underline{v} = \left(\frac{L}{m}\right)\left(\frac{1}{\alpha}\left(1+\epsilon\cos\theta\right)\underline{e}_\theta + \frac{\epsilon\sin\theta}{\alpha}\underline{e}_r\right)$$

$$\underline{v} = \left(\frac{L}{m\alpha}\right)\left(\left(1+\epsilon\cos\theta\right)\underline{e}_\theta + \epsilon\sin\theta\,\underline{e}_r\right). \quad -(12)$$

Therefore

$$v^2 = \left(\frac{L}{m\alpha}\right)^2\left(\left(1+\epsilon\cos\theta\right)^2 + \epsilon^2\sin^2\theta\right). \quad -(13)$$

$$v^2 = \left(\frac{L}{m\alpha}\right)^2\left(1+\epsilon^2+2\epsilon\cos\theta\right) \quad -(14)$$

The relativistic force law for any orbit is:

$$\underline{F} = -\frac{L^2}{mr^3}\left(r^2\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right)\underline{e}_r + \frac{L^4}{m^3r^3c^2}\frac{d}{d\theta}\left(\frac{1}{r}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\underline{e}_\theta \quad -(15)$$

3) The force can be expressed as a function of θ for any planar orbit using eqns. (9) and (15).

This is best left to computer algebra. One interesting case is the precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad - (16)$$

In these equations the angular momentum L is a constant of motion and can be coded in as a constant.

The velocity v can be expressed in terms of ω and r by using eqs. (10), (14) and:

$$L = m r^2 \omega . \quad - (17)$$

So:

$$v^2 = \left(\frac{r^2 \omega}{\alpha}\right)^2 \left(1 + \epsilon^2 + 2\epsilon \cos\theta\right) \quad - (18)$$

where

$$\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) . \quad - (19)$$

so

$$\boxed{v^2 = \frac{r^4 \omega^2}{\alpha^2}\left(1 + \epsilon^2 + 2\left(\frac{\alpha}{r} - 1\right)\right)} \quad - (20)$$

In the case of the circle:

4)

$$\epsilon = 0, \quad \alpha = r \quad -(21)$$

So

$$v = \omega r. \quad -(22)$$

For the general planar orbit:

$$\underline{v} = r^2 \omega \left(\frac{1}{r} \underline{e}_\theta - \frac{d}{d\theta}\left(\frac{1}{r}\right) \underline{e}_r \right) \quad -(23)$$

i.e

$$v^2 = r^2 \omega^2 + r^4 \omega^2 \left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2,$$

$$\boxed{v^2 = \omega^2 r^2 \left(1 + r^2 \left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 \right)} \quad -(24)$$

for any planar orbit.