238(1): Relativistic Kinematics

In this case the relativistic velocity is:

$$\underline{v} = \frac{d\underline{r}}{d\tau} = \gamma \frac{d\underline{r}}{dt} \quad -(1)$$

where τ is the proper time and γ the Lorentz factor:

$$\gamma = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2}, \quad -(2)$$

where c is the speed of light. The relativistic acceleration is

$$\underline{a} = \frac{d}{d\tau}\left(\frac{d\underline{r}}{d\tau}\right)$$

$$= \frac{d}{d\tau}\left(\gamma \frac{d\underline{r}}{dt}\right) \quad -(3)$$

$$= \gamma \frac{d}{dt}\left(\gamma \frac{d\underline{r}}{dt}\right)$$

Therefore

$$\underline{a} = \gamma \left(\frac{d\gamma}{dt} \frac{d\underline{r}}{dt} + \gamma \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right)\right) \quad -(4)$$

Denote

$$\underline{v} = \frac{d\underline{r}}{d\tau} \quad -(5)$$

then

$$\boxed{\underline{a} = \gamma \left(\frac{d\gamma}{dt} \underline{v} + \gamma \frac{d\underline{v}}{dt}\right)} \quad -(6)$$

2) The velocity appearing in the Lorentz factor is defined by:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(7)$$

where

$$d\underline{r} \cdot d\underline{r} = v_0^2 \, dt^2 \quad -(8)$$

so

$$c^2 d\tau^2 = \left(c^2 - v_0^2\right) dt^2 \quad -(9)$$

and

$$\frac{dt}{d\tau} = \gamma = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} \quad -(10)$$

in which

$$d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\theta^2 \quad -(11)$$

in plane polar coordinates. So the v appearing in the Lorentz transform is, from eqs. (8) and (11):

$$v_0^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(12)$$

$$= \underline{v}_0 \cdot \underline{v}_0$$

where

$$\underline{v}_0 = \frac{d\underline{r}}{dt}. \quad -(13)$$

therefore

$$\boxed{v_0 = \left|\underline{v}_0\right| = \left|\frac{d\underline{r}}{dt}\right|} \quad -(14)$$

3) In plane polar coordinates:

$$\underline{r} = r\underline{e}_r, \quad -(15)$$

and

$$\underline{v}_0 = \frac{d\underline{r}}{dt} = \frac{d}{dt}(r\underline{e}_r)$$

$$= \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt}$$

$$= \frac{dr}{dt}\underline{e}_r + \omega r\underline{e}_\theta \quad -(16)$$

$$= \frac{dr}{dt}\underline{e}_r + \underline{\omega}\times\underline{r}$$

$$= \left(\frac{L}{m}\right)\left(\frac{1}{r}\underline{e}_\theta - \frac{d}{d\theta}\left(\frac{1}{r}\right)\underline{e}_r\right)$$

is the non-relativistic velocity. Here:

$$L = mr^2\frac{d\theta}{dt} = mr^2\omega \quad -(17)$$

is the magnitude of the angular momentum:

$$\underline{L} = \underline{r}\times\underline{p} = m\underline{r}\times\underline{v}_0 \quad -(18)$$

__The relativistic velocity is:__

$$\boxed{\begin{aligned}\underline{v} &= \gamma\underline{v}_0 \quad -(19)\\ &= \left(1-\frac{v_0^2}{c^2}\right)^{-1/2}\underline{v}_0\end{aligned}}$$

The relativistic momentum is:

$$\underline{p} = m\underline{v} \quad -(20)$$

i.e.
$$\underline{p} = \gamma m \underline{v_0} = \gamma m \frac{d\underline{r}}{dt} = m \frac{d\underline{r}}{d\tau} \quad -(21)$$

From eq. (21):
$$p^2 c^2 = \gamma^2 m^2 c^4 \left(\frac{v_0}{c}\right)^2$$
$$= \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right) \quad -(22)$$
$$= \gamma^2 m^2 c^4 - m^2 c^4$$

So
$$\boxed{E^2 = c^2 p^2 + m^2 c^4} \quad -(23)$$

which is known as the Einstein energy equation, in which:
$$E = \gamma m c^2,$$
$$p = \gamma m v_0 \quad -(24)$$
$$E_0 = mc^2.$$

In plane polar coordinates:
$$v_0 = \left(\left(\frac{dr}{dt}\right)^2 + r^2 \omega^2\right)^{1/2}. \quad -(25)$$

This is as far as special relativity goes, the acceleration is not considered in the context of the Einstein energy equation, and plane polar coordinates are almost never considered in the usual development of special relativity. This situation is incomplete and somewhat self contradictory because the same theory defines the Minkowski force equation:

$$\underline{F} = m \frac{d\underline{v}}{d\tau} \quad -(26)$$

which involves acceleration. From eqs. (6) and (26):

$$\boxed{\underline{F} = m\underline{a} = \gamma m \left(\frac{d\gamma}{dt}\underline{v} + \gamma \frac{d\underline{v}}{dt} \right)} \quad -(27)$$

Another unsatisfactory feature of the usual development of special relativity is that plane polar coordinates are not used, or very rarely used. In fact, eq. (23) is easily derived from eqs. (7) and (11) as follows:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(28)$$

So

$$mc^2 = mc^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2$$

$$= \gamma^2 mc^2 - \left(\left(\frac{dr}{d\tau}\right)^2 + r^2 \left(\frac{d\theta}{d\tau}\right)^2 \right)$$

$$= \frac{E^2}{mc^2} - \frac{p^2}{c^2} \quad -(29)$$

So

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(30)$$

QED.

6) Therefore special relativity can be applied to rotational motion and orbital theory. The acceleration can be defined in special relativity and is:

$$\underline{a} = \frac{d}{d\tau}\left(\frac{d\underline{r}}{d\tau}\right) \quad -(31)$$

$$= \gamma \frac{d}{dt}\left(\gamma \frac{d\underline{r}}{dt}\right)$$

$$\boxed{\underline{a} = \gamma\left(\frac{d\gamma}{dt}\frac{d\underline{r}}{dt} + \gamma\frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right)\right)} \quad -(32)$$

In plane polar coordinates:

$$\frac{d\underline{r}}{dt} = \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r}, \quad -(33)$$

$$\frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) = \frac{d^2 r}{dt^2}\underline{e}_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(34)$$

It is seen that the relativistic acceleration contains more terms than the non-relativistic acceleration.