

238(14) : Relativistic Theory of the Circular Section
Orbits and Precessing Circular Sections

In this case :

$$\frac{d\theta}{d\tau} = \frac{L_0}{mr^2} \quad - (1)$$

where τ is the proper time and where the Lorentz factor

is :

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad - (2)$$

in which the velocity is defined by :

$$\underline{v} = \frac{L_0}{m} \left(\frac{1}{r} \underline{e}_\theta - \frac{d}{dt} \left(\frac{1}{r} \right) \underline{e}_r \right) \quad - (3)$$

From eq. (1) :

$$\begin{aligned} r^2 d\theta &= \left(1 - \frac{v^2}{c^2}\right)^{1/2} \frac{L_0}{m} dt \\ &= \frac{L_0}{m} d\tau = \frac{1}{\gamma} \frac{L_0}{m} dt. \end{aligned} \quad - (4)$$

Therefore :

$$dt = \frac{m}{L_0} \left(1 - \frac{v^2}{c^2}\right)^{-1/2} r^2 d\theta \quad - (5)$$

and

$$t = \frac{m}{L_0} \int \left(1 - \frac{v^2}{c^2}\right)^{-1/2} r^2 d\theta \quad - (6)$$

For the circular sections :

$$2) \quad r = \frac{d}{1 + \epsilon \cos \theta} \quad \text{--- (7)}$$

and as in note 238(4):

$$v^2 = \left(\frac{L_0}{md} \right)^2 (1 + \epsilon^2 + 2\epsilon \cos \theta) \quad \text{--- (8)}$$

So:

$$t = \frac{m}{L_0} \int \left(1 - \left(\frac{L_0}{md} \right)^2 (1 + \epsilon^2 + 2\epsilon \cos \theta) \right)^{-1/2} \frac{d^2}{(1 + \epsilon \cos \theta)^2} d\theta \quad \text{--- (9)}$$

This has to be integrated numerically.

The animation proceeds as in note 238(12).

using:

$$x = \frac{d \cos \theta}{1 + \epsilon \cos \theta}, \quad \text{--- (10)}$$

$$y = \frac{d \sin \theta}{1 + \epsilon \cos \theta}, \quad \text{--- (11)}$$

For each θ calculate t from eq. (9).

For example:

$$t_1 = \frac{m}{L_0} \int_0^{\theta_1} \left(1 - \left(\frac{L_0}{md} \right)^2 (1 + \epsilon^2 + 2\epsilon \cos \theta) \right)^{-1/2} \frac{d^2}{(1 + \epsilon \cos \theta)^2} d\theta \quad \text{--- (12)}$$

Repeat for t_2, t_3, \dots, t_n and animate
 (x, y) as a function of t .

Precessing Conical Sections

In this case:

$$r = \frac{d}{1 + \epsilon \cos(x\theta)} \quad - (13)$$

so

$$t = \frac{m}{L_0} \int \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \frac{d^2}{(1 + \epsilon \cos(x\theta))^2} d\theta \quad - (14)$$

From eqs. (3) and (13):

$$\underline{v} = \frac{L_0}{m} \left(\left(\frac{1 + \epsilon \cos(x\theta)}{d} \right) \underline{e}_\theta - \frac{d}{d\theta} \left(\frac{1}{r} \right) \underline{e}_r \right) \quad - (15)$$

where:

$$\frac{d}{d\theta} \left(\frac{1}{r} \right) = - \frac{x\epsilon \sin(x\theta)}{d} \quad - (16)$$

so:

$$v^2 = \left(\frac{L_0}{md} \right)^2 \left((1 + \epsilon \cos(x\theta))^2 + x^2 \epsilon^2 \sin^2(x\theta) \right) \quad - (17)$$

Therefore t is calculated from eqs. (14) and (17) for a given θ . For example:

$$t_1 = \frac{m}{L_0} \int_0^{\theta_1} \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \frac{d^2}{(1 + \epsilon \cos(x\theta))^2} d\theta \quad - (18)$$

$$v^2 = \left(\frac{L_0}{md} \right)^2 \left((1 + \epsilon \cos(x\theta))^2 + x^2 \epsilon^2 \sin^2(x\theta) \right) \quad - (19)$$