235(10): Spin Torsion Elements and Riemann Torsion of the Plane Polar Coordinates

In this note the spin torsion elements are worked out with:

$$(a) = 2. \quad -(1)$$

In this case:

$$T^{(2)}_{12} = \partial_1 q^{(2)}_2 - \partial_2 q^{(2)}_1 + \omega^{(2)}_{1(b)} q^{(b)}_2 - \omega^{(2)}_{2(b)} q^{(b)}_1 \quad -(2)$$

The basic unit vectors are:

$$\underline{e}_r = \underline{i} \cos\theta + \underline{j} \sin\theta \quad -(3)$$

$$\underline{e}_\theta = -\underline{i} \sin\theta + \underline{j} \cos\theta \quad -(4)$$

so
$$\frac{d\underline{e}_\theta}{dr} = -\frac{d\theta}{dr} \underline{e}_r \quad -(5)$$

and
$$\frac{d\underline{e}_\theta}{d\theta} = -\underline{e}_r \quad -(6)$$

so
$$\frac{d\underline{e}_\theta}{d(r\theta)} = -\left(\frac{d\theta}{dr}\right) \underline{e}_r \quad -(7)$$

Eq. (5) is:
$$\frac{de^{(2)}}{dx^1} = \omega^{(2)}_{1(1)} e^{(1)} \quad -(8)$$

and eq. (7) is:
$$\frac{de^{(2)}}{dx^2} = \omega^{(2)}_{2(1)} e^{(1)} \quad -(9)$$

So:

$$\boxed{\omega^{(2)}{}_{1(1)} = \omega^{(2)}{}_{2(1)} = -\frac{d\theta}{dr}} \quad -(10)$$

The tetrad is defined by:

$$q^{(a)}{}_{\mu} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad -(11)$$

So

$$q^{(1)}{}_{1} = \cos\theta \,, \quad q^{(1)}{}_{2} = \sin\theta \quad -(12)$$

$$q^{(2)}{}_{1} = -\sin\theta \,, \quad q^{(2)}{}_{2} = \cos\theta \,.$$

Therefore:

$$T^{(2)}{}_{12} = \partial_1 q^{(2)}{}_{2} - \partial_2 q^{(2)}{}_{1} + \omega^{(2)}{}_{1(1)} q^{(1)}{}_{2} - \omega^{(2)}{}_{2(1)} q^{(1)}{}_{1}$$

$$= \frac{d\cos\theta}{dr} + \frac{d}{d(r\theta)}\sin\theta - \frac{d\theta}{dr}\sin\theta + \frac{d\theta}{dr}\cos\theta$$

$$= 2\frac{d\theta}{dr}\left(\cos\theta - \sin\theta\right) . \quad -(13)$$

Similarly:

$$T^{(2)}{}_{12} = \partial_2 q^{(2)}{}_{1} - \partial_1 q^{(2)}{}_{2} + \omega^{(2)}{}_{2(1)} q^{(1)}{}_{1} - \omega^{(2)}{}_{2(1)} q^{(1)}{}_{2}$$

$$= -\frac{d}{d(r\theta)}\sin\theta - \frac{d\cos\theta}{dr} - \frac{d\theta}{dr}\cos\theta + \frac{d\theta}{dr}\sin\theta$$

$$= -2\frac{d\theta}{dr}\left(\cos\theta - \sin\theta\right) \quad -(14)$$

3) Therefore:

$$T^{(2)}_{\ 12} = -T^{(2)}_{\ 21} \quad -(15)$$

From the tetrad postulate:

$$\Gamma^{(a)}_{\ \mu\nu} = \partial_\mu q^{(a)}_{\ \nu} + \omega^{(a)}_{\ \mu\nu} \quad -(16)$$

so:

$$T^{(1)}_{\ 12} = \Gamma^{(1)}_{\ 12} - \Gamma^{(1)}_{\ 21} \quad -(17)$$

$$= 2\frac{d\theta}{dr}\left(\cos\theta + \sin\theta\right)$$

and

$$T^{(2)}_{\ 12} = \Gamma^{(2)}_{\ 12} - \Gamma^{(2)}_{\ 21} \quad -(18)$$

$$= 2\frac{d\theta}{dr}\left(\cos\theta - \sin\theta\right).$$

Therefore:

$$\Gamma^{(1)}_{\ 12} = 2\frac{d\theta}{dr}\cos\theta \quad -(19)$$

$$\Gamma^{(1)}_{\ 21} = -2\frac{d\theta}{dr}\sin\theta \quad -(20)$$

The <u>Riemann</u> <u>torsion</u> is given by:

$$T^{\kappa}_{\ \mu\nu} = \Gamma^{\kappa}_{\ \mu\nu} - \Gamma^{\kappa}_{\ \nu\mu} \quad -(21)$$

and is related to the Cartan torsion by the

4) tetrad postulate:

$$D_\nu q^a_\mu = 0 \quad -(22)$$

Therefore:

$$\Gamma^{(1)}_{12} = 2\frac{d\theta}{dr}\cos\theta, \quad \Gamma^{(1)}_{21} = -2\frac{d\theta}{dr}\sin\theta \quad -(23)$$

$$\Gamma^{(2)}_{12} = 2\frac{d\theta}{dr}\cos\theta, \quad \Gamma^{(2)}_{21} = 2\frac{d\theta}{dr}\sin\theta$$

so

$$\begin{bmatrix} \Gamma^{(1)}_{12} & \Gamma^{(1)}_{21} \\ \Gamma^{(2)}_{12} & \Gamma^{(2)}_{21} \end{bmatrix} = 2\frac{d\theta}{dr}\begin{bmatrix} \cos\theta & -\sin\theta \\ \cos\theta & \sin\theta \end{bmatrix} \quad -(24)$$

Therefore the torsion for the plane polar coordinate system are non-zero. From eqns. (23) the connections are not symmetric. Therefore the use of a symmetric connection in Einsteinian general relativity is incorrect.

In general:

$$\Gamma^{(a)}_{\mu\nu} = \frac{1}{2}\left(\Gamma^{(a)}_{\mu\nu}(S) + \Gamma^{(a)}_{\mu\nu}(A)\right) \quad -(25)$$

where S denotes symmetric and A denotes anti-symmetric in μ and ν. In eq. (25) the following definitions are used:

$$\Gamma^{(a)}_{\mu\nu}(S) = \Gamma^{(a)}_{\nu\mu}(S) \quad -(26)$$

$$\Gamma^{(a)}_{\mu\nu}(A) = -\Gamma^{(a)}_{\nu\mu}(A) \quad -(27)$$

so
$$T^{(a)}_{\mu\nu} = \frac{1}{2}\left(\Gamma^{(a)}_{\mu\nu}(A) - \Gamma^{(a)}_{\nu\mu}(A)\right) \quad -(28)$$

$$= \Gamma^{(a)}_{\mu\nu}(A)$$

It follows ~~that~~:

$$\Gamma^{(1)}_{12}(A) = 2\frac{d\theta}{dr}\left(\cos\theta + \sin\theta\right) \quad -(29)$$

$$\Gamma^{(2)}_{12}(A) = 2\frac{d\theta}{dr}\left(\cos\theta - \sin\theta\right)$$

In vector format:

$$\Gamma^{(1)}_{3} = \epsilon_{123}\Gamma^{(1)}_{12} \quad -(30)$$

and
$$\underline{\Gamma} = 2\frac{d\theta}{dr}\left(\cos\theta + \sin\theta\right)\underline{k} \quad -(31)$$

i.e
$$\boxed{\underline{\Gamma} = 2\left(\cos\theta + \sin\theta\right)\underline{\omega}} \quad -(32)$$