231(5): Notes Concerning the Fundamental Meaning of the Tetrad.

In order to isolate and define the tetrad it is necessary to use the following procedure, transverse components being used for the sake of illustration:

$$\begin{bmatrix} \underline{e}^{(1)} \\ \underline{e}^{(2)} \end{bmatrix} = \begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 \\ q^{(2)}_1 & q^{(2)}_2 \end{bmatrix} \begin{bmatrix} \underline{e}^1 \\ \underline{e}^2 \end{bmatrix} \quad -(1)$$

Multiply both side of eq. (1) by $[\underline{e}_1 \ \underline{e}_2]$ from the right:

$$\begin{bmatrix} \underline{e}^{(1)} \\ \underline{e}^{(2)} \end{bmatrix} \cdot [\underline{e}_1 \ \underline{e}_2] = \begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 \\ q^{(2)}_1 & q^{(2)}_2 \end{bmatrix} \begin{bmatrix} \underline{e}^1 \\ \underline{e}^2 \end{bmatrix} \cdot [\underline{e}_1, \underline{e}_2] \quad -(2)$$

so

$$\begin{bmatrix} \underline{e}^{(1)} \cdot \underline{e}_1 & \underline{e}^{(1)} \cdot \underline{e}_2 \\ \underline{e}^{(2)} \cdot \underline{e}_1 & \underline{e}^{(2)} \cdot \underline{e}_2 \end{bmatrix} = \begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 \\ q^{(2)}_1 & q^{(2)}_2 \end{bmatrix} \begin{bmatrix} \underline{e}^1 \cdot \underline{e}_1 & \underline{e}^1 \cdot \underline{e}_2 \\ \underline{e}^2 \cdot \underline{e}_1 & \underline{e}^2 \cdot \underline{e}_2 \end{bmatrix}$$

In the Cartesian basis for space:

$$\begin{bmatrix} \underline{e}^1 \cdot \underline{e}_1 & \underline{e}^1 \cdot \underline{e}_2 \\ \underline{e}^2 \cdot \underline{e}_1 & \underline{e}^2 \cdot \underline{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(3)$$

so

$$\begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 \\ q^{(2)}_1 & q^{(2)}_2 \end{bmatrix} = \begin{bmatrix} \underline{e}^{(1)} \cdot \underline{e}_1 & \underline{e}^{(1)} \cdot \underline{e}_2 \\ \underline{e}^{(2)} \cdot \underline{e}_1 & \underline{e}^{(2)} \cdot \underline{e}_2 \end{bmatrix} \quad -(4).$$

Therefore:

$$\boxed{\begin{aligned} q^a_\mu &= \underline{e}^a \cdot \underline{e}_\mu \quad &-(5) \\ \underline{e}^a &= q^a_\mu \, \underline{e}^\mu \quad &-(6) \end{aligned}}$$

The tetrad postulate is therefore:

$$D_\nu q^a_\mu = D_\nu \left(\underline{e}^a \cdot \underline{e}_\mu \right) = 0 \quad -(7)$$

i.e.

$$\boxed{D_\nu \underline{e}^a \cdot \underline{e}_\mu + \underline{e}^a \cdot D_\nu \underline{e}_\mu = 0} \quad -(8)$$

In extending this to the momentum:

$$\begin{bmatrix} \underline{p}^{(1)} \\ \underline{p}^{(2)} \end{bmatrix} = \begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 \\ q^{(2)}_1 & q^{(2)}_2 \end{bmatrix} \begin{bmatrix} \underline{p}^1 \\ \underline{p}^2 \end{bmatrix} \quad -(9)$$

where

$$\underline{p}^{(1)} = \frac{1}{\sqrt{2}} \left(p_x \underline{i} - i p_y \underline{j} \right) \quad -(10)$$

$$\underline{p}^{(2)} = \frac{1}{\sqrt{2}} \left(p_x \underline{i} + i p_y \underline{j} \right) \quad -(11)$$

$$\underline{p}^1 = p_x \underline{i} \quad -(12)$$

$$\underline{p}^2 = p_y \underline{j} \quad -(13)$$

From these equations:

$$\underline{p}^{(1)} = q^{(1)}_1 \underline{p}^1 + q^{(1)}_2 \underline{p}^2 \quad -(14)$$

$$\underline{p}^{(2)} = q^{(2)}_1 \underline{p}^1 + q^{(2)}_2 \underline{p}^2 \quad -(15)$$

So

$$\frac{1}{\sqrt{2}}\left(p_x \underline{i} - i p_y \underline{j}\right) = q^{(1)}_1 p_x \underline{i} + q^{(1)}_2 p_y \underline{j} \quad -(16)$$

$$\frac{1}{\sqrt{2}}\left(p_x \underline{i} + i p_y \underline{j}\right) = q^{(2)}_1 p_x \underline{i} + q^{(2)}_2 p_y \underline{j} \quad -(17)$$

Therefore

$$q^{(1)}_1 = \frac{1}{\sqrt{2}}, \quad q^{(1)}_2 = -\frac{i}{\sqrt{2}} \quad -(18)$$

$$q^{(2)}_1 = \frac{1}{\sqrt{2}}, \quad q^{(2)}_2 = \frac{i}{\sqrt{2}}. \quad -(19)$$

Multiply both sides of eq. (9) from the right by $[\underline{p}_1 \ \underline{p}_2]$, then:

$$\begin{bmatrix} \underline{p}^{(1)} \\ \underline{p}^{(2)} \end{bmatrix} \cdot [\underline{p}_1, \underline{p}_2] = \begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 \\ q^{(2)}_1 & q^{(2)}_2 \end{bmatrix} \begin{bmatrix} \underline{p}^1 \\ \underline{p}^2 \end{bmatrix} \cdot [\underline{p}_1 \ \underline{p}_2] \quad -(20)$$

then:

$$\begin{bmatrix} \underline{p}^{(1)} \cdot \underline{p}_1 & \underline{p}^{(1)} \cdot \underline{p}_2 \\ \underline{p}^{(2)} \cdot \underline{p}_1 & \underline{p}^{(2)} \cdot \underline{p}_2 \end{bmatrix} = \begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 \\ q^{(2)}_1 & q^{(2)}_2 \end{bmatrix} \begin{bmatrix} \underline{p}^1 \cdot \underline{p}_1 & \underline{p}^1 \cdot \underline{p}_2 \\ \underline{p}^2 \cdot \underline{p}_1 & \underline{p}^2 \cdot \underline{p}_2 \end{bmatrix} \quad -(21)$$

In the Cartesian basis:

4)
$$A = \begin{bmatrix} \underline{p}^1 \cdot \underline{p}_1 & \underline{p}^1 \cdot \underline{p}_2 \\ \underline{p}^2 \cdot \underline{p}_1 & \underline{p}^2 \cdot \underline{p}_2 \end{bmatrix} = \begin{bmatrix} p_x^2 & 0 \\ 0 & p_y^2 \end{bmatrix} \quad -(22)$$

The inverse matrix of eq. (22) is
$$A^{-1} = \frac{1}{p_x^2 p_y^2} \begin{bmatrix} p_y^2 & 0 \\ 0 & p_x^2 \end{bmatrix} \quad -(23)$$

So
$$\begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix} = \begin{bmatrix} \underline{p}^{(1)} \cdot \underline{p}_1 & \underline{p}^{(1)} \cdot \underline{p}_2 \\ \underline{p}^{(2)} \cdot \underline{p}_1 & \underline{p}^{(2)} \cdot \underline{p}_2 \end{bmatrix} A^{-1}$$

$$= \frac{1}{p_x^2 p_y^2} \begin{bmatrix} \underline{p}^{(1)} \cdot \underline{p}_1 & \underline{p}^{(1)} \cdot \underline{p}_2 \\ \underline{p}^{(2)} \cdot \underline{p}_1 & \underline{p}^{(2)} \cdot \underline{p}_2 \end{bmatrix} \begin{bmatrix} p_y^2 & 0 \\ 0 & p_x^2 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{p}^{(1)} \cdot \underline{p}_1 & \underline{p}^{(1)} \cdot \underline{p}_2 \\ \underline{p}^{(2)} \cdot \underline{p}_1 & \underline{p}^{(2)} \cdot \underline{p}_2 \end{bmatrix} \begin{bmatrix} 1/p_x^2 & 0 \\ 0 & 1/p_y^2 \end{bmatrix} \quad -(24)$$

i.e.
$$q_1^{(1)} = \frac{1}{p_x^2} \underline{p}^{(1)} \cdot \underline{p}_1 = \frac{1}{\sqrt{2}} \quad -(25)$$

$$q_2^{(1)} = \frac{1}{p_y^2} \underline{p}^{(1)} \cdot \underline{p}_2 = -\frac{i}{\sqrt{2}} \quad -(26)$$

$$q_1^{(2)} = \frac{1}{p_x^2} \underline{p}^{(2)} \cdot \underline{p}_1 = \frac{1}{\sqrt{2}} \quad -(27)$$

$$q_2^{(2)} = \frac{1}{p_y^2} \underline{p}^{(2)} \cdot \underline{p}_2 = \frac{i}{\sqrt{2}} \quad -(28)$$

Similarly: $$q_3^{(3)} = \frac{1}{p_z^2} \underline{p}^{(3)} \cdot \underline{p}_3 = 1 \quad -(29)$$

In 3-D Space:

$$p^2 = p_x^2 + p_y^2 + p_z^2 \quad -(30)$$

Therefore:

$$\underline{p}^{(1)} \cdot \underline{p}_1 + \underline{p}^{(2)} \cdot \underline{p}_2 + \underline{p}^{(3)} \cdot \underline{p}_3$$

$$= \frac{p_x^2}{\sqrt{2}} + i \frac{p_y^2}{\sqrt{2}} + p_z^2 \quad -(31)$$

$-(32)$

$$\boxed{\text{So: } p^2 = p_x^2 + p_y^2 + p_z^2 = \sqrt{2}\, \underline{p}^{(1)} \cdot \underline{p}_1 - i\sqrt{2}\, \underline{p}^{(2)} \cdot \underline{p}_2 + \underline{p}^{(3)} \cdot \underline{p}_3}$$

In the next note this analysis will be extended to the Einstein energy equation.