231(4): Further Development of the Einstein Energy Equation.

Consider the energy momentum four vector p^μ, then:

$$p^a = q^a_\mu p^\mu \quad -(1)$$

then

$$p^a p_\mu = q^a_\mu (p^\mu p_\mu) = m^2 c^2 q^a_\mu$$
$$= (q^a_\mu p^\mu) p_\mu \quad -(2)$$

Here:

$$p^\mu = \left(\frac{E}{c}, p^1, p^2, p^3\right), \quad -(3)$$
$$p^a = \left(\frac{E}{c}, p^{(1)}, p^{(2)}, p^{(3)}\right), \quad -(4)$$

So

$$p^0 = p^{(0)} = \frac{E}{c}. \quad -(5)$$

Written out in full, eq. (1) is: $\quad -(6)$

$$\begin{bmatrix} p^{(0)} \\ p^{(1)} \\ p^{(2)} \\ p^{(3)} \end{bmatrix} = \begin{bmatrix} q^{(0)}_0 & q^{(0)}_1 & q^{(0)}_2 & q^{(0)}_3 \\ q^{(1)}_0 & q^{(1)}_1 & q^{(1)}_2 & q^{(1)}_3 \\ q^{(2)}_0 & q^{(2)}_1 & q^{(2)}_2 & q^{(2)}_3 \\ q^{(3)}_0 & q^{(3)}_1 & q^{(3)}_2 & q^{(3)}_3 \end{bmatrix} \begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{bmatrix}$$

For the sake of illustration consider the complex circular basis, then

$$p^{(0)} = q^{(0)}_0 p^0 \quad -(7)$$
$$p^{(1)} = q^{(1)}_1 p^1 + q^{(1)}_2 p^2 \quad -(8)$$
$$p^{(2)} = q^{(2)}_1 p^1 + q^{(2)}_2 p^2 \quad -(9)$$
$$p^{(3)} = q^{(3)}_3 p^3 \quad -(10)$$

2) The Schroedinger postulate is:

$$p^\mu = i\hbar \partial^\mu \quad -(11)$$

where

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla} \right) \quad -(12)$$

$$p^\mu = \left(p^0, \underline{p} \right). \quad -(13)$$

Therefore the quantum equivalents of eqs. (7) to (10) are as follows:

$$\hat{p}^{(0)}\phi = i\hbar \frac{\partial \phi}{\partial t} \quad -(14)$$

$$\hat{p}^{(3)}\phi = -i\hbar \frac{\partial \phi}{\partial Z} \quad -(15)$$

$$\hat{p}^{(1)}\phi = q^{(1)}_1 \hat{p}^1 \phi + q^{(1)}_2 \hat{p}^2 \phi \quad -(16)$$

$$\hat{p}^{(2)}\phi = q^{(2)}_1 \hat{p}^1 \phi + q^{(2)}_2 \hat{p}^2 \phi. \quad -(17)$$

So:

$$\hat{p}^{(1)}\phi = -\frac{i\hbar}{\sqrt{2}} \left(\frac{\partial \phi}{\partial X} - i\frac{\partial \phi}{\partial Y} \right) \quad -(18)$$

$$\hat{p}^{(2)}\phi = -\frac{i\hbar}{\sqrt{2}} \left(\frac{\partial \phi}{\partial X} + i\frac{\partial \phi}{\partial Y} \right) \quad -(19)$$

The transverse tetrad elements are evaluated with

$$\begin{bmatrix} \underline{e}^{(1)} \\ \underline{e}^{(2)} \end{bmatrix} = \begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 \\ q^{(2)}_1 & q^{(2)}_2 \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} \quad -(20)$$

where:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} \underline{i} - i\underline{j} \\ \underline{i} + i\underline{j} \end{bmatrix} = \begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix}\begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix}, \quad -(21)$$

So:

$$q_1^{(1)} = \frac{1}{\sqrt{2}}, \quad q_2^{(1)} = \frac{-i}{\sqrt{2}} \quad -(22)$$

$$q_1^{(2)} = \frac{1}{\sqrt{2}}, \quad q_2^{(2)} = \frac{i}{\sqrt{2}}. \quad -(23)$$

At second order:

$$\hat{p}^a \hat{p}_\mu \phi = m^2 c^2 q_\mu^a \phi \quad -(24)$$

where

$$\hat{p}_\mu \phi = i\hbar \partial_\mu \phi \quad -(25)$$

and

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla}\right). \quad -(26)$$

Therefore

$$\hat{p}_0 \phi = i\frac{\hbar}{c}\frac{\partial \phi}{\partial t} \quad -(27)$$

$$\hat{p}_1 \phi = -i\hbar \frac{\partial \phi}{\partial X} \quad -(28)$$

$$\hat{p}_2 \phi = -i\hbar \frac{\partial \phi}{\partial Y} \quad -(29)$$

$$\hat{p}_3 \phi = -i\hbar \frac{\partial \phi}{\partial Z}. \quad -(30)$$

4)

Therefore:

$$\hat{p}_0 \phi = \frac{i\hbar}{c}\frac{\partial\phi}{\partial t}, \quad \hat{p}^{(0)}\phi = i\hbar\frac{\partial\phi}{\partial t},$$

$$\hat{p}_1 \phi = -i\hbar\frac{\partial\phi}{\partial X}, \quad \hat{p}^{(1)}\phi = -\frac{i\hbar}{\sqrt{2}}\left(\frac{\partial\phi}{\partial X} - i\frac{\partial\phi}{\partial Y}\right),$$

$$\hat{p}_2 \phi = -i\hbar\frac{\partial\phi}{\partial Y}, \quad \hat{p}^{(2)}\phi = -\frac{i\hbar}{\sqrt{2}}\left(\frac{\partial\phi}{\partial X} + i\frac{\partial\phi}{\partial Y}\right),$$

$$\hat{p}_3 \phi = -i\hbar\frac{\partial\phi}{\partial Z}, \quad \hat{p}^{(3)}\phi = -i\hbar\frac{\partial\phi}{\partial Z},$$

$$\hat{p}^0 \phi = \frac{i\hbar}{c}\frac{\partial\phi}{\partial t}, \quad \hat{p}_{(0)}\phi = i\hbar\frac{\partial\phi}{\partial t},$$

$$\hat{p}^1 \phi = -i\hbar\frac{\partial\phi}{\partial X}, \quad \hat{p}_{(1)}\phi = -\frac{i\hbar}{\sqrt{2}}\left(\frac{\partial\phi}{\partial X} - i\frac{\partial\phi}{\partial Y}\right)$$

$$\hat{p}^2 \phi = -i\hbar\frac{\partial\phi}{\partial Y}, \quad \hat{p}_{(2)}\phi = -\frac{i\hbar}{\sqrt{2}}\left(\frac{\partial\phi}{\partial X} + i\frac{\partial\phi}{\partial Y}\right)$$

$$\hat{p}^3 \phi = -i\hbar\frac{\partial\phi}{\partial Z}, \quad \hat{p}_{(3)}\phi = -i\hbar\frac{\partial\phi}{\partial Z}. \qquad -(31)$$

Now note the result:

$$\underline{e}^{(1)}\cdot\underline{e}^{(2)} + \underline{e}^{(2)}\cdot\underline{e}^{(1)} = \frac{1}{2}\left(\underline{i} - i\underline{j}\right)\cdot\left(\underline{i} + i\underline{j}\right)$$

$$+ \frac{1}{2}\left(\underline{i} + i\underline{j}\right)\cdot\left(\underline{i} - i\underline{j}\right) \qquad -(32)$$

$$= \underline{i}\cdot\underline{i} + \underline{j}\cdot\underline{j}.$$

5) It follows that:

$$p^0 p_0 + p^1 p_1 + p^2 p_2 + p^3 p_3 = m^2 c^2 \quad -(33)$$

$$p^{(0)} p_{(0)} + p^{(1)} p_{(2)} + p^{(2)} p_{(1)} + p^{(3)} p_{(3)} = m^2 c^2, \quad -(34)$$

$$\left(\hat{p}^0 \hat{p}_0 + \hat{p}^1 \hat{p}_1 + \hat{p}^2 \hat{p}_2 + \hat{p}^3 \hat{p}_3 \right) \phi = -\hbar^2 \Box \phi$$
$$= m^2 c^2 \phi \quad -(35)$$

$$\left(\hat{p}^{(0)} \hat{p}_{(0)} + \hat{p}^{(1)} \hat{p}_{(2)} + \hat{p}^{(2)} \hat{p}_{(1)} + \hat{p}^{(3)} \hat{p}_{(3)} \right) \phi$$
$$= -\hbar^2 \Box \phi = m^2 c^2 \phi \quad -(36).$$

Eqs. (35) and (36) give the Klein-Gordon equation:

$$\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) \phi = 0 \quad -(37)$$

using two different representations of the same Minkowski spacetime. These are the Cartesian:

$$\underline{i} \times \underline{j} = \underline{k} \quad -(38)$$
$$\underline{k} \times \underline{i} = \underline{j}$$
$$\underline{j} \times \underline{k} = \underline{i}$$

and circular polar:

$$\underline{e}^{(1)} \times \underline{e}^{(2)} = i\,\underline{e}^{(3)*}$$

$$\underline{e}^{(3)} \times \underline{e}^{(1)} = i\,\underline{e}^{(2)*} \quad -(39)$$

$$\underline{e}^{(2)} \times \underline{e}^{(3)} = i\,\underline{e}^{(1)*}$$

where * means complex conjugate. Here:

$$\underline{e}^{(1)} = \underline{e}^{(2)*} = \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j}) \quad -(40)$$

$$\underline{e}^{(2)} = \underline{e}^{(1)*} = \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j}) \quad -(41)$$

$$\underline{e}^{(3)} = \underline{e}^{(3)*} = \underline{k}. \quad -(42)$$

In order to describe a circularly polarized wave of spacetime propagating along the $\underline{k}$ axis, the (1), (2), (3) frame is made to rotate and translate with respect to the 1, 2, 3 frame. This is defined as follows:

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j})\,e^{i\phi} \quad -(43)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j})\,e^{-i\phi} \quad -(44)$$

where the phase is defined by:

$$\phi = \omega t - \kappa Z \quad -(45)$$

Here ω is the angular frequency at instant t, and κ the wave-vector at Z.

7) The circularly polarized electromagnetic potential is defined by:

$$\underline{A}^{(1)} = A^{(0)} \underline{e}^{(1)} \quad - (46)$$

$$\underline{A}^{(2)} = A^{(0)} \underline{e}^{(2)} \quad - (47)$$

$$\underline{A}^{(3)} = A^{(0)} \underline{e}^{(3)} \quad - (48)$$

The momentum associated with the frame of reference (43) and (44) is:

$$\underline{p}^{(1)} = p^{(0)} \underline{e}^{(1)} \quad - (49)$$

$$\underline{p}^{(2)} = p^{(0)} \underline{e}^{(2)} \quad - (50)$$

$$\underline{p}^{(3)} = p^{(0)} \underline{e}^{(3)} \quad - (51)$$

The presence of the phase factors $e^{i\phi}$ and $e^{-i\phi}$ makes no difference to the cyclical symmetry of eq. (39), and no difference to the Einstein energy equation (36) and Klein Gordon equation (37). This is because the phase factors cancel in the products $\hat{p}^{(1)}\hat{p}^{(2)}$ and $\hat{p}^{(2)}\hat{p}^{(1)}$, and in the cross products $\underline{e}^{(1)} \times \underline{e}^{(2)}$ and $\underline{e}^{(2)} \times \underline{e}^{(1)}$. However, the individual momentum vectors $\underline{p}^{(1)}$ and $\underline{p}^{(2)}$ are phase dependent, because the frame (1), (2), (3) itself is phase dependent.

Eq. (30) is modified to:

$$\begin{bmatrix} \underline{e}^{(1)} e^{i\phi} \\ \underline{e}^{(2)} e^{-i\phi} \end{bmatrix} = \begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} \quad -(52)$$

i.e.:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} (\underline{i} - i\underline{j}) e^{i\phi} \\ (\underline{i} + i\underline{j}) e^{-i\phi} \end{bmatrix} = \begin{bmatrix} q_1^{(1)} & q_2^{(1)} \\ q_1^{(2)} & q_2^{(2)} \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} \quad -(53)$$

so:

$$q_1^{(1)} = \frac{1}{\sqrt{2}} e^{i\phi}, \quad q_2^{(1)} = \frac{-i}{\sqrt{2}} e^{i\phi},$$
$$q_1^{(2)} = \frac{1}{\sqrt{2}} e^{-i\phi}, \quad q_2^{(2)} = \frac{i}{\sqrt{2}} e^{i\phi}. \quad -(54)$$

The tetrad elements become phase dependent, indicating that the theory has become one of general relativity.

More generally, the components of momentum in the Cartesian representation are given by:

$$p^1 = p_x, \; p^2 = p_y, \; p^3 = p_z \quad -(55)$$
$$p_1 = -p_x, \; p_2 = -p_y, \; p_3 = -p_z$$

in Minkowski spacetime.

1) In circular polar representation:

$$p^{(1)} = \frac{1}{\sqrt{2}}\left(p_X - i p_Y\right), \quad p^{(2)} = \frac{1}{\sqrt{2}}\left(p_X + i p_Y\right),$$

$$p_{(1)} = -\frac{1}{\sqrt{2}}\left(p_X - i p_Y\right), \quad p_{(2)} = -\frac{1}{\sqrt{2}}\left(p_X + i p_Y\right). \quad -(56).$$

So:

$$p^{(1)} p^{(2)} + p^{(2)} p^{(1)} = p_X^2 + p_Y^2, \quad -(57)$$

and

$$p^{(1)} p_{(2)} + p^{(2)} p_{(1)} = -\left(p_X^2 + p_Y^2\right). \quad -(58)$$

It follows that:

$$p^0 p_0 + p^1 p_1 + p^2 p_2 + p^3 p_3 = m^2 c^2 \quad -(59)$$

and

$$p^{(0)} p_{(0)} + p^{(1)} p_{(2)} + p^{(2)} p_{(1)} + p^{(3)} p_{(3)} = m^2 c^2 \quad -(60)$$

which again are two representations of the Einstein energy equation in Minkowski spacetime, i.e. special relativity.

In general relativity:

$$p^{(1)} = \frac{1}{\sqrt{2}}\left(p_X - i p_Y\right) e^{i\phi} \quad -(61)$$

$$p^{(2)} = \frac{1}{\sqrt{2}}\left(p_X + i p_Y\right) e^{-i\phi} \quad -(62)$$

which is an elliptically polarized wave

10) of linear momentum. Eqs. (61) and (62) define the contravariant components. The covariant components are:

$$p_{(1)} = -\frac{1}{\sqrt{2}}\left(p_x - i p_y\right) e^{i\phi} \quad -(63)$$

$$p_{(2)} = -\frac{1}{\sqrt{2}}\left(p_x + i p_y\right) e^{-i\phi} \quad -(64)$$

Again it is seen that the phases do not affect eq. (62).

However, the phases enter into the analysis through equations such as:

$$\hat{p}^{(1)} \hat{p}_1 \phi = p^{(1)} p_1 \phi. \quad -(65)$$

Here:

$$p^{(1)} = \frac{1}{\sqrt{2}}\left(p_x - i p_y\right) e^{i\phi} \quad -(66)$$

$$p_1 = - p_x \quad -(67)$$

and so:

$$\hat{p}^{(1)} = -\frac{i\hbar}{\sqrt{2}}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) e^{i\phi} \quad -(67)$$

$$\hat{p}_1 = -i\hbar\frac{\partial}{\partial X} \quad -(68)$$

Define:

$$\phi_1 = e^{i\phi}\phi \quad -(69)$$

11) then:

$$-\frac{\hbar^2}{\sqrt{2}}\left(\frac{\partial}{\partial X} - i\frac{\partial}{\partial Y}\right)\frac{\partial \phi_1}{\partial X} \qquad -(70)$$

$$= -\frac{1}{\sqrt{2}}\left(p_X - i p_Y\right) p_X \phi_1$$

i.e. $$\hbar^2\left(\frac{\partial^2}{\partial X^2} - i\frac{\partial}{\partial Y}\frac{\partial}{\partial X}\right)\phi_1 = \left(p_X^2 - i p_Y p_X\right)\phi_1 \qquad -(71)$$

The complex conjugate of this equation is:

$$\hbar^2\left(\frac{\partial^2}{\partial X^2} + i\frac{\partial}{\partial Y}\frac{\partial}{\partial X}\right)\phi_1^* = \left(p_X^2 + i p_Y p_X\right)\phi_1^* . \qquad -(72)$$

In order to complete this analysis note that

$$p^2 = p_X^2 + p_Y^2 + p_Z^2 \qquad -(73)$$

$$= \underline{p}^{(3)}\cdot\underline{p}^{3} + \sqrt{2}\left(\underline{p}^{(1)}\cdot\underline{p}^{1} - i\,\underline{p}^{(2)}\cdot\underline{p}^{2}\right)$$

and $$\frac{E^2}{c^2} - p^2 = m^2c^2 \qquad -(74)$$

In this notation:

$$\underline{p}^{1} = p_x \underline{i}, \quad \underline{p}^{2} = p_y \underline{j}, \quad \underline{p}^{3} = p_z \underline{k} \qquad -(75)$$

$$\underline{p}^{(1)} = \frac{1}{\sqrt{2}} \left(p_x \underline{i} - i p_y \underline{j} \right) \qquad -(76)$$

$$\underline{p}^{(2)} = \frac{1}{\sqrt{2}} \left(p_x \underline{i} + i p_y \underline{j} \right) \qquad -(77)$$

$$\underline{p}^{(3)} = p_z \underline{k} \qquad -(78)$$

Eq. (74) is the Einstein energy equation of special relativity. It becomes an equation of <u>general</u> relativity when:

$$\underline{p}^{(1)} \rightarrow \underline{p}^{(1)} e^{i\phi} \qquad -(79)$$

$$\underline{p}^{(2)} \rightarrow \underline{p}^{(2)} e^{-i\phi}. \qquad -(80)$$

Eqs. (79) and (80) may be regarded as a <u>transformation</u>, under which m^2c^2 remains <u>invariant</u>. Thus:

$$p^2 \rightarrow \underline{p}^{(3)} \cdot \underline{p}^{3} + \sqrt{2} \left(\underline{p}^{(1)} \cdot \underline{p}^{1} e^{i\phi} - i \underline{p}^{(2)} \cdot \underline{p}^{2} e^{-i\phi} \right) \qquad -(81)$$

$$= p_z^2 + p_x^2 e^{i\phi} + p_y^2 e^{-i\phi}$$

Denote:

$$\pi^2 = p_z^2 + p_x^2 e^{i\phi} + p_y^2 e^{-i\phi} \quad -(82)$$

Since m^2c^2 is invariant, then:

$$E \rightarrow \epsilon \quad -(83)$$

and

$$\boxed{\frac{\epsilon^2}{c^2} - \pi^2 = m^2c^2} \quad -(84)$$

It is seen that the frame of reference defined by eqs. (43) and (44) has resulted in eq. (84) of general relativity.

The next stage in the calculation is to quantize eq. (84) to give a generalized Klein Gordon equation, an equation of combined quantum mechanics and general relativity. This method has been illustrated with the circular polar basis but can be used with any basis. Eq. (84) gives the effect of frame movement on the Einstein energy equation. The dynamics of spacetime itself are incorporated.

4)

To end this, note the quantization process can be investigated with component quantization such as:

$$\hat{p}^{(1)} \phi = p^{(1)} \phi. \quad -(85)$$

where:

$$p^{(1)} = \frac{1}{\sqrt{2}} (p_X - i p_Y). \quad -(86)$$

Apply the Schroedinger postulates to eq. (86) to obtain $\hat{p}^{(1)}$:

$$p_X = -i\hbar \frac{\partial}{\partial X} \quad -(87)$$

$$p_Y = -i\hbar \frac{\partial}{\partial Y} \quad -(88)$$

so

$$\hat{p}^{(1)} = -\frac{i\hbar}{\sqrt{2}} \left(\frac{\partial}{\partial X} - i \frac{\partial}{\partial Y} \right) \quad -(89)$$

Therefore eq. (85) is:

$$-i\hbar \left(\frac{\partial}{\partial X} - i \frac{\partial}{\partial Y} \right) \phi = (p_X - i p_Y) \phi$$

$$-(90)$$

The next note will develop this quantization technique.