231(a): Proof of Eq (16) of Note 231(1) and of Commutativity of Triple Matrix Products.

To Prove

$$(AB)C = A(BC) \qquad -(1)$$

for matrices.

Proof

Consider:

$$\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\right)\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

$$= \begin{bmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{bmatrix}\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

$$= \begin{bmatrix} A_1B_1C_1 + A_2B_3C_1 + A_1B_2C_3 + A_2B_4C_3 & \cdots \\ \cdots & \cdots \end{bmatrix} \qquad -(2)$$

Secondly consider:

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\left(\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}\right)$$

$$= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} B_1C_1 + B_2C_3 & B_1C_2 + B_2C_4 \\ B_3C_1 + B_4C_3 & B_3C_2 + B_4C_4 \end{bmatrix} \qquad -(3)$$

$$= \begin{bmatrix} A_1B_1C_1 + A_1B_2C_3 + A_2B_3C_1 + A_2B_4C_3 & \cdots \\ \cdots & \cdots \end{bmatrix}$$

It is seen that the products are the same, QED

The same matrix method is used in tensorial

2) algebra. For example:

$$V^{a} = q^{a}_{\mu} V^{\mu} \quad -(4)$$

which in two transverse dimensions is:

$$\begin{bmatrix} V^{(1)} \\ V^{(2)} \end{bmatrix} = \begin{bmatrix} q^{(1)}_{1} & q^{(1)}_{2} \\ q^{(2)}_{1} & q^{(2)}_{2} \end{bmatrix} \begin{bmatrix} V^{1} \\ V^{2} \end{bmatrix} \quad -(5)$$

$$= \begin{bmatrix} q^{(1)}_{1} V^{1} + q^{(1)}_{2} V^{2} \\ q^{(2)}_{1} V^{1} + q^{(2)}_{2} V^{2} \end{bmatrix}$$

It is seen that there is summation over the repeated μ indices. This is equivalent to matrix algebra.

Next consider:

$$g^{a\nu} = q^{a}_{\mu} g^{\mu\nu} = \begin{bmatrix} q^{(1)}_{1} & q^{(1)}_{2} \\ q^{(2)}_{1} & q^{(2)}_{2} \end{bmatrix} \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} \quad -(6)$$

$$= \begin{bmatrix} q^{(1)}_{1} g^{11} + q^{(1)}_{2} g^{21} & q^{(1)}_{1} g^{12} + q^{(1)}_{2} g^{22} \\ q^{(2)}_{1} g^{11} + q^{(2)}_{2} g^{21} & q^{(2)}_{1} g^{12} + q^{(2)}_{2} g^{22} \end{bmatrix}$$

In each of the four elements there is summation over the repeated μ index.

Next consider:

$$g^{a\nu} g_{\mu\nu} = q^{a}_{\mu} g^{\mu\nu} g_{\mu\nu} \quad -(7)$$

3)

To Prove

$$\left(q^{a}_{\mu} g^{\mu\nu}\right) g_{\mu\nu} = q^{a}_{\mu}\left(g^{\mu\nu} g_{\mu\nu}\right) \quad -(8)$$

Proof

The left hand side is:

$$LHS = \begin{bmatrix} q^{(1)}_{1} g^{11} + q^{(1)}_{2} g^{21} & q^{(1)}_{1} g^{12} + q^{(1)}_{2} g^{22} \\ q^{(2)}_{1} g^{11} + q^{(2)}_{2} g^{21} & q^{(2)}_{1} g^{12} + q^{(2)}_{2} g^{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

$$= \begin{bmatrix} q^{(1)}_{1} g^{11} g_{11} + q^{(1)}_{2} g^{21} g_{11} + q^{(1)}_{1} g^{12} g_{21} + q^{(1)}_{2} g^{22} g_{21} \cdots \\ \cdots \end{bmatrix} \quad -(9)$$

The right hand side is:

$$RHS = \begin{bmatrix} q^{(1)}_{1} & q^{(1)}_{2} \\ q^{(2)}_{1} & q^{(2)}_{2} \end{bmatrix} \left(\begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \right)$$

$$= \begin{bmatrix} q^{(1)}_{1} & q^{(1)}_{2} \\ q^{(2)}_{1} & q^{(2)}_{2} \end{bmatrix} \begin{bmatrix} g^{11} g_{11} + g^{12} g_{21} & g^{11} g_{12} + g^{12} g_{22} \\ g^{21} g_{11} + g^{22} g_{21} & g^{21} g_{12} + g^{22} g_{22} \end{bmatrix}$$

$$= \begin{bmatrix} q^{(1)}_{1} g^{11} g_{11} + q^{(1)}_{1} g^{12} g_{21} + q^{(1)}_{2} g^{21} g_{11} + q^{(1)}_{2} g^{22} g_{21} \cdots \\ \cdots \end{bmatrix} \quad -(10)$$

4) It is seen that the first element of the LHS and RHS products are the same, Q.E.D.

Eq. (15) of note 231(1) is:

$$g^{a\nu} = q^a_{\ \mu} \, g^{\mu\nu} \quad -(11)$$

indicating that:

$$\boxed{q^a_{\ \mu} = g^a_{\ \mu}} \quad -(12)$$

This is a fundamental result indicating that the tetrad is a mixed index metric.

In eq. (11) $g^{\mu\nu}$ is the inverse metric in any mathematical space of any dimension. The metric is $g_{\mu\nu}$. So:

$$g^{\mu\nu} g_{\mu\nu} = 1 \quad -(13)$$

Multiply both sides of eq. (11) by $g_{\mu\nu}$:

$$g^{a\nu} g_{\mu\nu} = q^a_{\ \mu} \, g^{\mu\nu} g_{\mu\nu} \quad -(14)$$

$$= q^a_{\ \mu} \left(g^{\mu\nu} g_{\mu\nu} \right)$$

$$= q^a_{\ \mu}$$

5) So:

$$\boxed{q^a_\mu = g^a_\mu = g^{a\nu} g_{\mu\nu}} \quad -(15)$$

which is eq. (16) of note 231(1), QED.

The metric $g^{a\nu}$ is g^a_μ with raised index, and in tensor algebra in any space of any dimension the metric $g_{\mu\nu}$ or inverse metric $g^{\mu\nu}$ is always used to lower or raise indices.

A particularly useful example of eq (15) is when μ and ν label the Minkowski spacetime. Here

$$g_{\mu\nu} = g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(16)$$

then:

$$q^a_\mu = g^a_\mu = g^{a\nu} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(17)$$

Let the a space be labelled by:

$$a = (0), (1), (2), (3) \quad -(18)$$

) Then:

$$g^{a\nu} = \begin{bmatrix} g^{(0)0} & g^{(0)1} & g^{(0)2} & g^{(0)3} \\ g^{(1)0} & g^{(1)1} & g^{(1)2} & g^{(1)3} \\ g^{(2)0} & g^{(2)1} & g^{(2)2} & g^{(2)3} \\ g^{(3)0} & g^{(3)1} & g^{(3)2} & g^{(3)3} \end{bmatrix} \quad -(19)$$

and

$$q^a_\mu = g^a_\mu = \begin{bmatrix} g^{(0)}_0 & g^{(0)}_1 & g^{(0)}_2 & g^{(0)}_3 \\ g^{(1)}_0 & g^{(1)}_1 & g^{(1)}_2 & g^{(1)}_3 \\ g^{(2)}_0 & g^{(2)}_1 & g^{(2)}_2 & g^{(2)}_3 \\ g^{(3)}_0 & g^{(3)}_1 & g^{(3)}_2 & g^{(3)}_3 \end{bmatrix} \quad -(20)$$

$$= \begin{bmatrix} g^{(0)0} & -g^{(0)1} & -g^{(0)2} & -g^{(0)3} \\ g^{(1)0} & -g^{(1)1} & -g^{(1)2} & -g^{(1)3} \\ g^{(2)0} & -g^{(2)1} & -g^{(2)2} & -g^{(2)3} \\ g^{(3)0} & -g^{(3)1} & -g^{(3)2} & -g^{(3)3} \end{bmatrix} \quad -(21)$$

In general:

$$\boxed{g^{a\nu} = h^a h^\nu \, \underline{e}^a \cdot \underline{e}^\nu} \quad -(22)$$

where

$$h^a = |\underline{e}^a| \quad -(23)$$

$$h^\nu = |\underline{e}^\nu| \quad -(24)$$

7)

So:

$$\boxed{q^a_\mu = \gamma^a_\mu = h^a h_\mu \, \underline{e}^a \cdot \underline{e}_\mu} \quad -(25)$$

where:

$$h^a = |\underline{e}^a| , \; h_\mu = |\underline{e}_\mu| . \quad -(26)$$

In Minkowski spacetime:

$$e^\sim = (\underline{1}, \underline{i}, \underline{j}, \underline{k}) \quad -(27)$$

In the spacetime labelled by (a):

$$e^a = (\underline{1}, \underline{e}^{(1)}, \underline{e}^{(2)}, \underline{e}^{(3)}) . \quad -(28)$$

Therefore $$e_\mu = (\underline{1}, -\underline{i}, -\underline{j}, -\underline{k}) . \quad -(29)$$

In the Minkowski spacetime:

$$|\underline{i}| = |\underline{j}| = |\underline{k}| = 1 , \quad -(30)$$

so $$h^0 = h^1 = h^2 = h^3 = 1 \quad -(31)$$

If the a basis is chosen with unit scaling factors then:

$$\boxed{q^a_\mu = \gamma^a_\mu = \underline{e}^a \cdot \underline{e}_\mu} \quad -(32)$$

This is a fundamental definition that may be used throughout differential geometry.

8) Similarly:

$$\boxed{g^{a\mu} = \underline{e}^{a} \cdot \underline{e}^{\mu}} \quad -(33)$$

The Electromagnetic Potential

This was defined in the early development of ECE theory as:

$$A^a_\mu = A_0 \, q^a_\mu \quad -(34)$$

so

$$A^a_\mu = A_0 g^a_\mu = A_0 \underline{e}^{a} \cdot \underline{e}_{\mu}$$

$$\boxed{A^a_\mu = \underline{A}^{a} \cdot \underline{e}_{\mu}} \quad -(35)$$

where

$$\underline{A}^{a} = A_0 \underline{e}^{a}, \quad -(36)$$

i.e.

$$A^{(0)} = A_0 \quad -(37)$$

$$\underline{A}^{(i)} = A_0 \underline{e}^{(i)}, \quad i = 1, 2, 3 \quad -(38)$$

This definition was introduced originally to deal with circularly polarized radiation, in which

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i\phi} \quad -(39)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{-i\phi} \quad -(40)$$

$$\phi = \omega t - \kappa Z. \quad -(41)$$

9) Here ω is angular frequency at instant t and $\underline{\kappa}$ is the wave vector at locality $\underline{r}$ along the Z axis. So the a frame is rotating and translating with respect to the μ frame.

Equally well, eq. (35) can be written as:

$$\boxed{A_\mu^a = A_0 q_\mu^a = \underline{e}^{(a)} \cdot \underline{A}_\mu} \quad -(42)$$

This equation makes it clear that the e/m potential at the ECE level is defined from the potential at the U(1) level by a dot product with the unit vector $\underline{e}^{(a)}$.

For example:

$$A_0^{(0)} = e^{(0)} A_0 = A_0 \quad -(43)$$

$$A_1^{(1)} = \underline{e}^{(1)} \cdot \underline{A}_1$$

$$= -\underline{e}^{(1)} \cdot A_0 \underline{i}, \quad -(44)$$

$$A^{(1)1} = A_0 \underline{e}^{(1)} \cdot \underline{i} = A_X^{(1)}$$

$$A^{(1)2} = A_0 \underline{e}^{(1)} \cdot \underline{j} = A_Y^{(1)}$$

$$A^{(3)3} = A_0 \underline{e}^{(3)} \cdot \underline{k} = A_Z^{(3)}$$

$$-(45)$$

In general:

$$\boxed{A^{a\mu} = A_0 \, g^{a\mu} = \underline{e}^{(a)} \cdot \underline{A}^{\mu}} \quad -(46)$$

where:

$$\underline{e}^{(a)} = \left(1, \underline{e}^{(1)}, \underline{e}^{(2)}, \underline{e}^{(3)}\right) \quad -(47)$$

$$\underline{A}^{\mu} = \left(A_0, A_0 \underline{i}, A_0 \underline{j}, A_0 \underline{k}\right) \quad -(48)$$

and by definition:

$$\underline{e}^{(0)} \cdot \underline{A}^{0} = A_0 . \quad -(49)$$

More generally:

$$\underline{A}^{\mu} = \left(A_0, A_X \underline{i}, A_Y \underline{j}, A_Z \underline{k}\right). \quad -(50)$$

Advances over Concepts by Elie Cartan

Cartan introduced the concept of tetrad to define a tangent Minkowski spacetime at point P to a base manifold. His purpose was to define spinors in the general mathematical space. ECE theory applied the Cartan method to define the electromagnetic potential in circular polarization. This became a concept of general relativity, i.e. of a propagating frame labelled a spinning and translating with regard to a Minkowski frame μ.

11) In this note, eqs. (42) and (46) make it clear that ECE contains more information than U(1), it is introduced by forming a metric out of $\underline{e}^{(a)}$ and A^{μ}. In U(1) the a space is not considered at all, and only one representation used of spacetime. By defining A^{a}_{μ} as a mixed index metric, the structure equations of Maurer and Cartan can be used to define the torsion and curvature. On the U(1) level these concepts do not exist in electrodynamics. The tetrad postulate can be used to find the spin connection. The first structure equation gives the relation between field and potential, and the Cartan identity between torsion and curvature gives the field equations of electrodynamics. The connection becomes an intrinsic part of electrodynamics, and a generally covariant unified field theory emerges.

Eqs. (42) and (46) show that all the dynamic information is contained in the moving frame with unit vector $\underline{e}^{(a)}$.
