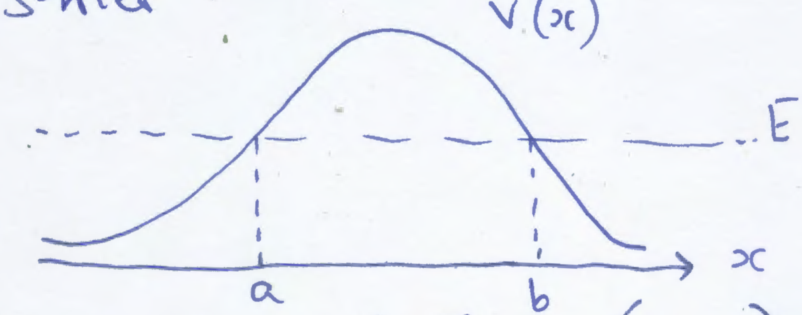
22a(1): Tunnelling Through an Arbitrary Barrier.

Consider the barrier:

Fig(1)



Using the Wentzel, Kramers, Brillouin (WKB) approx., then as in Merzbacher pp. 126 ff:

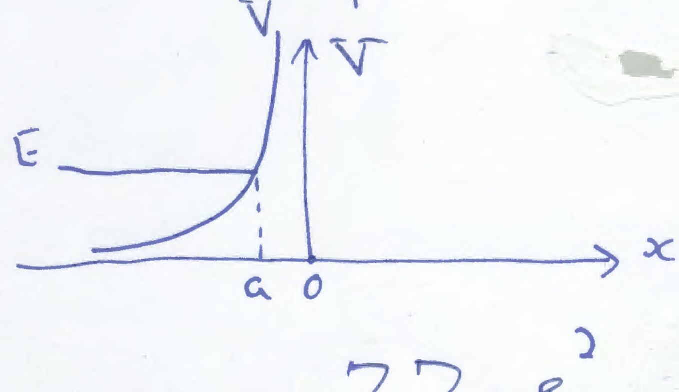
$$T = 4 \Big/ \left(2\theta + \frac{1}{2\theta} \right)^2 \quad -(1)$$

where:

$$\theta = \exp\left(\int_a^b \kappa(x)\,dx \right). \quad -(2)$$

Consider a nucleus of charge $Z_1 e$ fusing with a nucleus of charge $Z_2 e$. In order to do this the Coulomb barrier has to be quantum tunnelled:

Fig(2)



Then:

$$V = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 x} \quad -(3)$$

for

$$x < 0. \quad -(4)$$

2) The point a is defined by:

$$E = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 a} \quad -(5)$$

and it is assumed that

$$b = 0 . \quad -(6)$$

Then:

$$K = \left(\frac{2m}{\hbar}\right)^{1/2}\left(V - E\right)^{1/2} \quad -(7)$$

$$\int_a^0 K\,dx = \frac{(2mE)^{1/2}}{\hbar}\int_a^0 \left(\frac{a}{x} - 1\right)^{1/2} dx \quad -(8)$$

for $x < a$ $-(9)$

Therefore

$$\theta = \exp\left(\frac{(2mE)^{1/2}}{\hbar}\int_a^0 \left(\frac{a}{x} - 1\right)^{1/2} dx\right) \quad -(10)$$

and

$$T = \frac{4}{\left(2\theta + \frac{1}{2\theta}\right)^2}, \quad -(11)$$

$$E = Z_1 Z_2 e^2 / a . \quad -(12)$$

Therefore T depends on the mass m of the incoming atom, on Z_1, Z_2 and a as follows:

3)

$$\theta = \exp\left(\frac{e}{\hbar}\left(\frac{2m Z_1 Z_2}{4\pi\epsilon_0 a}\right)^{1/2} \int_a^0 \left(\frac{a}{x} - 1\right)^{1/2} dx \right) \quad -(13)$$

Here:

$$e = 1.60219 \times 10^{-19}\ C$$
$$\hbar = 6.62618 \times 10^{-34}\ Js$$
$$4\pi\epsilon_0 = 1.11265 \times 10^{-10}\ J^{-1}C^2 m^{-1}$$

Therefore:

$$\theta = \exp\left(\left(\frac{e^2 m Z_1 Z_2}{2\pi\hbar^2 \epsilon_0 a}\right)^{1/2} \int_a^0 \left(\frac{a}{x} - 1\right)^{1/2} dx \right) \quad -(14)$$

<u>Units Check</u>

$$\left(\frac{e^2 m Z_1 Z_2}{2\pi\hbar^2\epsilon_0 a}\right)^{1/2} = \left(\frac{C^2\, kgm}{J^2 s^2 J^{-1} C^2 m^{-1} m}\right)^{1/2} = \left(\frac{kgm}{Js^2}\right)^{1/2},$$

$$J = kgm\, m^2 s^{-2}$$

so the exponent is dimensionless as required because the units of the integral are m.

Now define: $\theta = e^{x_1}$ — (15)

then

$$T = \frac{4}{\left(2e^{x_1} + \frac{1}{2}e^{-x_1}\right)^2} \quad -(16)$$

4) where:

$$x_1 = \left(\frac{e^2 m Z_1 Z_2}{2\pi \hbar^2 \epsilon_0 a}\right)^{1/2} \int_a^0 \left(\frac{a}{x} - 1\right)^{1/2} dx \quad -(17)$$

$$x_1 = \left(\frac{e^2}{2\pi \hbar^2 \epsilon_0}\right)^{1/2} \left(\frac{m Z_1 Z_2}{a}\right)^{1/2} \int_a^0 \left(\frac{a}{x} - 1\right)^{1/2} dx \quad -(18)$$

The fine structure constant is:

$$\alpha = \frac{e^2}{4\pi \hbar c \epsilon_0} \quad -(19)$$

so

$$x_1 = \left(\frac{2\alpha c}{\hbar}\right)^{1/2} \left(\frac{m Z_1 Z_2}{a}\right)^{1/2} \int_a^0 \left(\frac{a}{x} - 1\right)^{1/2} dx \quad -(20)$$

where m is the mass of the proton. So:

$$x_1 = \left(\frac{2\alpha c m}{\hbar}\right)^{1/2} \left(\frac{Z_1 Z_2}{a}\right)^{1/2} \int_a^0 \left(\frac{a}{x} - 1\right)^{1/2} dx \quad -(21)$$

where

$\alpha = 0.007297351$ $-(22)$

$c = 2.997925 \times 10^{8}\ ms^{-1}$

$m = 1.67265 \times 10^{-27}\ kg$

$\hbar = 1.05459 \times 10^{-34}\ Js$

5)

Therefore:

$$\boxed{x_1 = 8.33 \times 10^6 \left(\frac{Z_1 Z_2}{a} \right)^{1/2} \int_a^0 \left(\frac{a}{x} - 1 \right)^{1/2} dx} \quad -(23)$$

Here

$$E = \frac{Z_1 Z_2}{4\pi\epsilon_0 a^2}, \quad -(24)$$

so

$$\frac{Z_1 Z_2}{a} = 4\pi\epsilon_0 a E, \quad -(25)$$

and

$$x_1 = 8.33 \times 10^6 \left(4\pi\epsilon_0 a E \right)^{1/2} \int_a^0 \left(\frac{a}{x} - 1 \right)^{1/2} dx$$

$$\boxed{x_1 = 87.9 \left(aE \right)^{1/2} \int_a^0 \left(\frac{a}{x} - 1 \right)^{1/2} x} \quad -(26)$$

and

$$T = \frac{4}{\left(2e^{x_1} + \frac{1}{2} e^{-x_1} \right)^2} \quad -(27)$$

For small a and E, T becomes large, and fusion is possible.