216(9): Some Comments on the Transition from Conical Section to Hyperbolic Spiral.

The condition for ~~this~~ transition is:

$$1 + \epsilon \cos (x\theta) = x\theta \quad -(1)$$

whose solution is

$$\theta = \theta_0, \quad -(2)$$

and

$$y = x\theta_0 \quad -(3)$$

If x is a constant then y is a constant and r is a constant for given α and ϵ. However if x is a variable, then y is a variable, and:

$$r = \frac{\alpha}{1 + \epsilon \cos \phi} = \frac{\alpha}{y} \quad -(4)$$

is a variable:

$$r = x(r). \quad -(5)$$

In general:

$$x = x(r, \theta) \quad -(6)$$

and

$$1 + \epsilon \cos \left(x(r, \theta)\, \theta \right) = x(r, \theta)\, \theta \quad -(7)$$

i.e

$$\boxed{x(r, \theta) = \frac{1}{\theta}\left(1 + \epsilon \left(\theta\, x(r, \theta) \right) \right)} \quad -(8)$$

This is a transcendental equation and the method can be supplemented by the following

2) Firstly the general hyperbola is reduced to the rectangular hyperbola. The general hyperbola is:

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1 \quad -(9)$$

where:

$$X = -a\epsilon + r\cos\theta \quad -(10)$$

$$Y = r\sin\theta \quad -(11)$$

$$r = -\epsilon X - a \quad -(12)$$

$$\epsilon = \left(1 + \frac{b^2}{a^2}\right)^{1/2} \quad -(13)$$

$$d = \frac{b^2}{a} \quad -(14)$$

From eqs. (10) and (12):

$$r = \frac{d}{1 + \epsilon\cos\theta} \quad -(15)$$

with

$$\epsilon > 1. \quad -(16)$$

The rectangular hyperbola is defined by:

$$a = b, \quad -(17)$$

i.e

$$\epsilon = \sqrt{2} \quad (18)$$

therefore:

$$r = -\sqrt{2}\,X - a \quad (19)$$

so

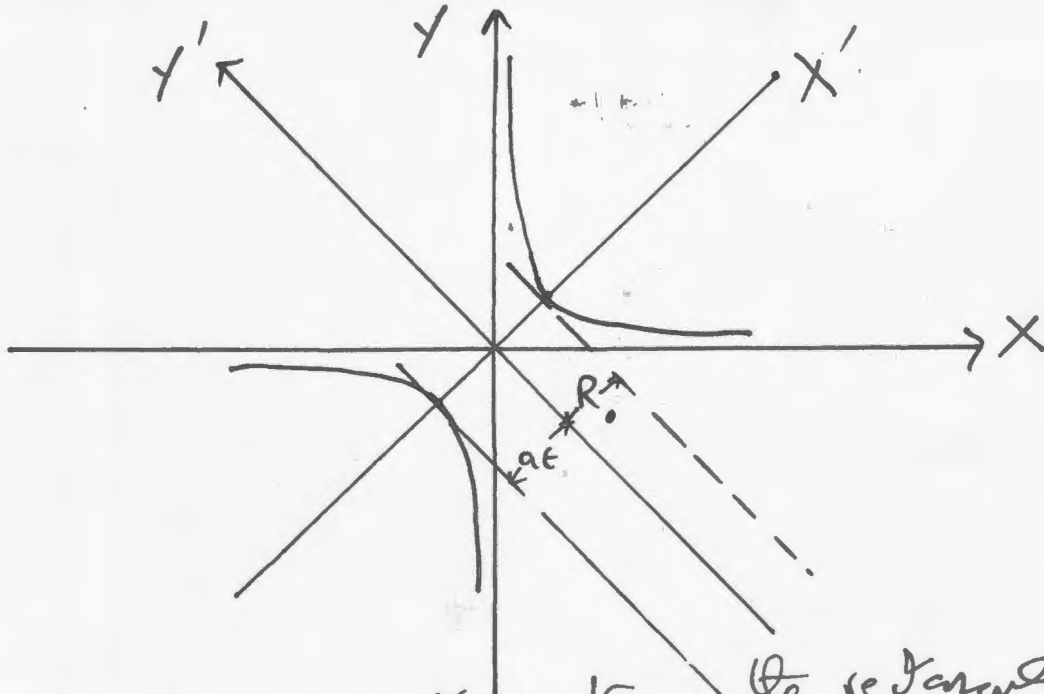
$$X^2 = \frac{(r-a)^2}{2}, \quad Y^2 = \frac{(r-a)^2}{2} - a^2 \quad -(20)$$

i.e.

$$X^2 + Y^2 = (r-a)^2 - a^2 \quad -(21)$$

With reference to the following figure:

5)



then in the YX coordinate system the rectangular hyperbola is

$$XY = R_0^2/2 \quad (22)$$

where the closest approach is defined by:

$$X = Y = R_0/\sqrt{2} \quad (23)$$

i.e.

$$R_0 = (X^2 + Y^2)^{1/2} \quad (24)$$

In the $Y'X'$ coordinate system the distance of closest approach is:

$$R_0 = \epsilon a = \sqrt{2}\, a \quad (25)$$

From eqns. (22) and (25):

$$\boxed{XY = a^2} \quad (26)$$

In general:

$$a = \frac{\alpha}{\epsilon^2 - 1} \quad (27)$$

From eqns. (18) and (27):

$$a = \alpha, \quad (28)$$

so

$$\boxed{XY = a^2 = \alpha^2 \text{ for } \epsilon = \sqrt{2}} \quad (29)$$

4)

Now define:

$$X = -\frac{r}{\sqrt{2}} \quad -(30)$$

from eq. (19). Then

$$rY = -\sqrt{2}\,a^2. \quad -(31)$$

Finally make the coordinate transformation:

$$\boxed{Y \rightarrow -\sqrt{2}\,a\phi} \quad -(31)$$

to obtain the hyperbolic spiral:

$$\boxed{r\phi = a} \quad -(32)$$

Under these conditions the hyperbola:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(33)$$

becomes the hyperbolic spiral:

$$r\phi = a = \alpha \quad -(34)$$

If

$$Y \rightarrow \sqrt{2}\,a\phi \quad -(35)$$

then

$$r\phi = -a. \quad -(36)$$

The angle ϕ is:

$$\boxed{\phi = \pm\frac{1}{\sqrt{2}}\frac{Y}{a}} \quad -(37)$$

and

$$\boxed{r = -\sqrt{2}X} \quad -(38)$$

From eqs. (12) and (31):

$$-a = r - \epsilon X \quad -(39)$$
$$= r - \sqrt{2}X$$

So $$\boxed{\phi = \frac{1}{\sqrt{2}}\left(\frac{Y}{r - \sqrt{2}X}\right)} \quad -(40)$$

Under these conditions the conical section can describe the orbits of spiral galaxies.