216(4): Deflection of the Precessing Hyperbola.

The precessing hyperbola is defined by:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(1)$$

where

$$\epsilon = \frac{1}{\sin(x\theta)} \quad -(2)$$

and

$$\alpha = R_0(1+\epsilon). \quad -(3)$$

Here (r, θ) are the plane cylindrical polar coordinates, α the semi right magnitude and R_0 the distance of closest approach, ϵ is the ellipticity, x is the precession constant.

In the small angle limit:

$$\sin(x\theta) \sim x\theta \quad -(4)$$

the deflection of the orbit is:

$$\Delta x\theta = 2\sin(x\theta) \quad -(5)$$

$$\sim x\theta = \frac{2}{\epsilon}$$

So the observable change in angle is:

$$* \quad \boxed{\Delta\theta = \frac{2}{\epsilon x}} \quad -(6)$$

As in note 215(7) the velocity of the hyperbolic orbit of particle of mass m is a

2) type (1) is:

$$v^2 = \left(\frac{L}{m\alpha}\right)^2 \left[\frac{2x^2\alpha}{r} - x^2(1-\epsilon^2) + \frac{\alpha^2}{r^2}(1-x^2)\right] \quad -(7)$$

and as in note 315(5):

$$\alpha = \frac{L_1^2}{m k_1}, \quad \epsilon = \left(1 + \frac{2EL_1^2}{m k_1^2}\right)^{1/2}, \quad -(8)$$

$$k_1 = x^2 k, \quad L_1^2 = L^2 - mk\alpha(1-x^2) \quad -(9)$$

These equations come from the lagrangian:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + \dot{\theta}^2 r^2\right) + x^2\frac{k}{r} + \frac{(1-x^2)\alpha k}{2r^2}. \quad -(10)$$

The lagrangian theory gives:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2}F(r) \quad -(11)$$

so with eq. (1):

$$F(r) = \frac{L^2}{m}\left(\frac{(x^2-1)}{r^3} - \frac{x^2}{\alpha r^2}\right). \quad -(12)$$

From eqs. (2) and (8):

$$\epsilon = \frac{1}{\sin(x\theta)} = \left(1 + \frac{2EL_1^2}{x^2 m^3 M^2 G^2}\right)^{1/2} \quad -(13)$$

$$\boxed{\sin(x\theta) = \left(1 + \frac{2EL_1^2}{x^2 m^3 M^2 G^2}\right)^{-1/2}} \quad -(14)$$

3) For small angles:

$$\theta = \frac{1}{x}\left(1 + \frac{2EL_1^2}{x^2 m^3 M^2 G^2}\right)^{-1/2} \quad -(15)$$

and the deflection is:

$$\Delta\theta = 2\theta = \frac{2}{x}\left(1 + \frac{2EL_1^2}{x^2 m^3 M^2 G^2}\right)^{-1/2} \quad -(16)$$

If the very rough Newtonian approximation is made:

$$E \sim \frac{1}{2}mc^2 \quad -(17)$$

then

$$\Delta\theta = \frac{2}{x}\left(1 + y^2\right)^{-1/2} \quad -(18)$$

$$= \frac{2}{x}\left(1 + y\right)^{-1}$$

where

$$y = \frac{cL_1}{xmMG} \quad -(19)$$

where from eq. (9):

$$L_1^2 = L^2 - m^2 MG\alpha\left(1 - x^2\right) - (20)$$

If

$$y \gg 1 \quad -(21)$$

$$\Delta\theta \sim \frac{2mMG}{cL_1} \quad -(22)$$

4) In the Newtonian theory:

$$L_1 = L = mcR_0 \quad -(23)$$

In the new theory:

$$\boxed{\Delta\theta = \frac{2mG}{c\left(c^2R_0^2 - mG\alpha(1-x^2)\right)^{1/2}}} \quad -(24)$$

At the distance of closest approach:

$$\cos(x\theta) = 0 \quad -(25)$$

so

$$\alpha = R_0(1+\epsilon) \quad -(26)$$

In eq. (24):

$$\alpha = \frac{L_1^2}{x^2 m^2 mG} \quad -(27)$$

Assume

$$L_1 \sim L = mcR_0 \quad -(28)$$

then:

$$\alpha \sim \frac{c^2R_0^2}{x^2 mG} \quad -(29)$$

and

$$\Delta\theta = \frac{2mG}{c^2R_0}\left(1 - \frac{(1-x^2)}{x^2}\right)^{-1/2} \quad -(30)$$

The final result is:

5)

$$\boxed{\Delta\theta \sim \left(\frac{x}{(2x^2-1)^{1/2}}\right)\frac{2MG}{c^2 R_0}} \quad -(31)$$

The experimental result is obtained with

$$\frac{x}{(2x^2-1)^{1/2}} = 2 \quad -(32)$$

i.e.

$$x^2 = 4(2x^2-1) \quad -(33)$$

$$7x^2 = 1, \quad x = \frac{1}{\sqrt{7}},$$

$$\boxed{x = 0.378}$$