1)

177(7): Force Eigenvalues for Particle on a Sphere

In this case the relevant equation is:

$$(\hat{H} - E)\,\underline{\nabla}\phi = \underline{F}\phi \quad -(1)$$

where:

$$\underline{\nabla}\phi = \frac{\partial\phi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underline{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\phi}\underline{e}_\phi \quad -(2)$$

and

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \quad -(3)$$

and

$$E = \frac{\hbar^2}{2mr^2}\,l(l+1). \quad -(4)$$

The eigenfunction of are the spherical harmonics.

Therefore:

$$\phi_{00} = \frac{1}{2\pi^{1/2}} \quad -(5)$$

and

$$F_{00} = 0. \quad -(6)$$

Secondly:

$$\phi_{10} = \frac{1}{2}\left(\frac{3}{\pi}\right)^{1/2}\cos\theta = A\cos\theta \quad -(7)$$

2) In this case:

$$\underline{\nabla}\phi_{10} = \frac{1}{r}\frac{\partial \phi_{10}}{\partial \theta}\,\underline{e}_\theta$$

$$= -\frac{A}{r}\sin\theta\,\underline{e}_\theta \qquad -(8)$$

and

$$\hat{H}\left(\underline{\nabla}\phi_{10}\right) = -\frac{\hbar^2}{2mr^3}\left(\frac{\cos\theta}{\sin\theta}\frac{\partial^2\phi_{10}}{\partial\theta^2} + \frac{\partial^3\phi_{10}}{\partial\theta^3}\right)\underline{e}_\theta$$

$$= \frac{\hbar^2 A}{2mr^3}\left(\frac{\cos^2\theta}{\sin\theta} + \sin\theta\right)\underline{e}_\theta, \qquad -(9)$$

with

$$E\,\underline{\nabla}\phi_{10} = -\frac{\hbar^2 A}{mr^3}\sin\theta\,\underline{e}_\theta. \qquad -(10)$$

So:

$$(\hat{H}-E)\underline{\nabla}\phi_{10} = \underline{F}\,\phi_{10} = \frac{\hbar^2 A}{2mr^3}\left(\frac{\cos^2\theta}{\sin\theta} + 3\sin\theta\right)\underline{e}_\theta \qquad -(11)$$

$$= A\cos\theta\,\underline{e}_\theta$$

So:

$$\boxed{F_{10} = \frac{\hbar^2}{2mr^3}\left(\frac{\cos\theta}{\sin\theta} + 3\frac{\sin\theta}{\cos\theta}\right)} \qquad -(12)$$

This is a pure quantum force, because the
assical force is zero, there being no potential