13(2): The g Factor and Spin-Orbit Coupling in the Standard Representation of Dirac.

This is:

$$(E - e\phi)\phi^R - c\,\underline{\sigma}\cdot(\underline{p} - e\underline{A})\phi^L = mc^2\phi^R \quad -(1)$$

$$-(E - e\phi)\phi^L + c\,\underline{\sigma}\cdot(\underline{p} - e\underline{A})\phi^R = mc^2\phi^L \quad -(2)$$

where the Dirac spinor is:

$$\phi = \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} \quad -(3)$$

From eq. (2):

$$\phi^L = \frac{c\,\underline{\sigma}\cdot\underline{\pi}}{E + mc^2 - e\phi}\phi^R \quad -(4)$$

where

$$\underline{\pi} = \underline{p} - e\underline{A}. \quad -(5)$$

In the non-relativistic approximation:

$$E = \gamma mc^2 \rightarrow mc^2 \quad -(6)$$

so:

$$\phi^L \sim \frac{c\,\underline{\sigma}\cdot\underline{\pi}}{2mc^2 - e\phi}\phi^R \quad -(7)$$

$$= \frac{\underline{\sigma}\cdot\underline{\pi}}{2mc^2\left(1 - \frac{e\phi}{2mc^2}\right)}\phi^R \sim \frac{\underline{\sigma}\cdot\underline{\pi}}{2mc}\left(1 + \frac{e\phi}{2mc^2}\right)\phi^R \quad -(8)$$

From eq. (8) in eq. (4):

$$E\phi^R = H\phi^R \quad -(9)$$

where

$$H = mc^2 + e\phi + \frac{1}{2m}\underline{\sigma}\cdot\underline{\pi}\left(1 + \frac{e\phi}{2mc^2}\right)\underline{\sigma}\cdot\underline{\pi} \quad -(10)$$

2) There are four terms:

$$H = mc^2 + e\phi + \frac{1}{2m}\underline{\sigma}\cdot\underline{\Pi}\,\underline{\sigma}\cdot\underline{\Pi} + \frac{1}{4m^2c^2}\underline{\sigma}\cdot\underline{\Pi}\,\phi\,\underline{\sigma}\cdot\underline{\Pi} \quad -(11)$$

The g factor is given by:

$$\frac{1}{2m}\left(\underline{\sigma}\cdot\underline{\Pi}\,\underline{\sigma}\cdot\underline{\Pi}\right)\phi^R = \left(\frac{(p-eA)^2}{2m} - \frac{e\hbar}{2m}\underline{\sigma}\cdot\underline{B}\right)\phi^R \quad -(12)$$

and the spin orbit coupling from the fourth term in eq. (11). In this term, if $\underline{A}$ is neglected for simplicity:

$$\underline{\sigma}\cdot\underline{p}\,\phi\,\underline{\sigma}\cdot\underline{p} = \frac{\hbar}{i}\underline{\sigma}\cdot\underline{\nabla}\,\phi\,\underline{\sigma}\cdot\underline{p}$$

$$= \frac{\hbar}{i}\left(\underline{\sigma}\cdot\left(\underline{\nabla}\phi\,\underline{\sigma}\cdot\underline{p} + \underline{\sigma}\cdot\underline{p}\,\underline{\nabla}\phi\right)\right)$$

$$= \frac{\hbar}{i}\underline{\sigma}\cdot\underline{E}\,\underline{\sigma}\cdot\underline{p} - \hbar^2\underline{\sigma}\cdot\underline{\nabla}\,\underline{\sigma}\cdot\underline{\nabla} \quad -(13)$$

The first term on the RHS of eq (13) is the spin-orbit coupling term and the second is the Darwin term. So:

$$(\underline{\sigma}\cdot\underline{E})(\underline{\sigma}\cdot\underline{p}) = \underline{E}\cdot\underline{p} + i\,\underline{\sigma}\cdot\underline{E}\times\underline{p}$$

and

$$H_{so} = \frac{\hbar}{4m^2c^2}\underline{\sigma}\cdot\underline{E}\times\underline{p} \quad -(14)$$

and eq. (14) gives the correct Thomas factor of 2.