1) 155(2): Coulomb Law from Spacetime and Metric, Equivalence of Curvature and Minimal Prescription.

Consider the interaction of the charge e_1 with the four potential:

$$A^\mu = \left(\frac{\phi}{c}, \underline{A} \right) \quad -(1)$$

In the minimal prescription, the interaction is:

$$p^\mu \rightarrow p^\mu + e_1 A^\mu \quad -(2)$$

where

$$p^\mu = \left(\frac{E}{c}, \underline{p} \right) \quad -(3)$$

So

$$E \rightarrow E + e_1 \phi \quad -(4)$$

$$\underline{p} \rightarrow \underline{p} + e_1 \underline{A} \quad -(5)$$

It is shown in this note that the interaction is the same as changing the metric.

Consider the hamiltonian:

$$H = \frac{1}{2m} p^\mu p_\mu = \frac{1}{2} \frac{E^2}{mc^2} - \frac{p^2}{2m} \quad -(6)$$

where:

$$E = mc^2 \frac{dt}{d\tau}, \quad -(7)$$

$$p^2 = \frac{1}{2} m \left(\frac{dr}{d\tau} \right)^2 + \frac{1}{2} \frac{L^2}{mr^2} \quad -(8)$$

The interaction of e_1 with A^μ results in:

$$H = \frac{1}{2m} \left(p^\mu + e_1 A^\mu \right) \left(p_\mu + e_1 A_\mu \right) \quad -(9)$$

$$= \frac{1}{2m} \left(\frac{E_1^2}{c^2} - p_1^2 \right)$$

2) where:

$$E_1 = \left(1 - \frac{r_0}{r}\right) E, \quad -(10)$$

and

$$\frac{p_1^2}{2m} = \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 \left(1 - \frac{r_0}{r}\right)^{-1} + \frac{1}{2}\frac{L^2}{mr^2} \quad -(11)$$

If:

$$\frac{r_0}{r} \ll 1, \quad -(12)$$

$$\left(1 - \frac{r_0}{r}\right)^{-1} \sim 1 + \frac{r_0}{r} \quad -(13)$$

So:

$$E \rightarrow E\left(1 - \frac{r_0}{r}\right) = E + e_1 \phi \quad -(14)$$

i.e.

$$\boxed{\frac{e_1 \phi}{E} = -\frac{r_0}{r}} \quad -(15)$$

From note 155(1):

$$r_0 = \frac{2 e_1 e_2}{4\pi\epsilon_0 m c^2} \quad -(16)$$

From eqs. (11) and (13):

$$\frac{p_1^2}{2m} = \frac{p^2}{2m} + \frac{1}{2} m \frac{r_0}{r}\left(\frac{dr}{d\tau}\right)^2 \quad -(17)$$

i.e.

$$\boxed{p_1^2 = p^2 + \frac{r_0}{r} p_r^2} \quad -(18)$$

where:

$$p_r = m \frac{dr}{d\tau} \quad -(19)$$

Refs:

3)

$$(p + e_1 A)^2 = p^2 + \frac{r_0}{r} p_r^2 \quad -(20)$$

If, for simplicity only, it is assumed that:

$$L = 0 \quad -(21)$$

then

$$p = p_r \quad -(22)$$

and

$$(p + e_1 A)^2 = p^2 \left(1 + \frac{r_0}{r}\right) \quad -(23)$$

If, finally, $e_1 A \ll p$ $\quad -(24)$

$$2 e_1 A p = p^2 \frac{r_0}{r}$$

$$\boxed{\frac{r_0}{r} = \frac{2 e_1 A}{p}} \quad -(25)$$

So the minimal prescription and charge /
metric are equivalent methods, QED.

The Coulomb law is produced from the metric

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 \left(1 - \frac{r_0}{r}\right) - dr^2 \left(1 - \frac{r_0}{r}\right)^{-1} - r^2 d\phi^2 \quad -(26)$$

with

$$\boxed{\frac{r_0}{r} = -\frac{e_1 \phi}{E} = \frac{2 e_1 A}{p} = \frac{2 e_1 e_2}{4\pi \epsilon_0 m c^2 r}} \quad -(27)$$