155(1): Coulomb and Newton Laws from Spacetime, Conservation of Energy and Momentum.

In this note it is shown that in general relativity, the Coulomb and Newton laws are properties of geometry, and in the derivation, H, E and L are conserved. These are the hamiltonian, total energy and angular momentum.

Start with the general metric of spherical spacetime:

$$ds^2 = c^2 d\tau^2 = e^{-r_0/r} c^2 dt^2 - e^{r_0/r} dr^2 - r^2 d\phi^2 \quad -(1)$$

The distance r_0 is defined as:

$$r_0 = 2\left(\frac{MG}{c^2} + \frac{e_1 e_2}{4\pi\epsilon_0 m c^2}\right) \quad -(2).$$

Here $d\tau$ is the infinitesimal of proper time, c is the speed of light in a vacuum, M is the mass of a particle that attracts the particle m, e_2 is the charge of the particle M, e_1 is the charge of the particle m, ϵ_0 is the vacuum permittivity, and G is Newton's constant.

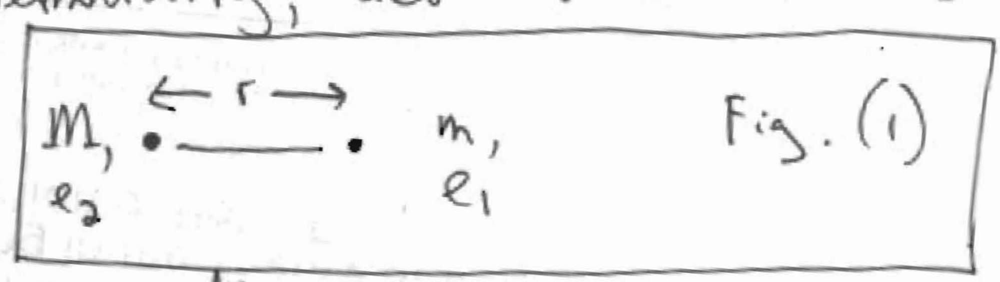

Fig. (1)

In the approximation:

$$e^{-r_0/r} \sim 1 - \frac{r_0}{r} \quad -(3)$$

eq. (1) becomes:

2)

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_0}{r}\right) c^2 dt^2 - \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 - r^2 d\phi^2 \quad -(4)$$

The Lagrangian is defined as:

$$H = \frac{1}{2} mc^2 = \frac{1}{2} m \left(\left(1 - \frac{r_0}{r}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{r_0}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\tau}\right)^2 \right) \quad -(5)$$

and is conserved.

The total energy E is a constant of motion and is also conserved, as is the angular momentum L.

$$E = mc^2 \left(1 - \frac{r_0}{r}\right) \left(\frac{dt}{d\tau}\right) \quad (6)$$

$$L = mr^2 \frac{d\phi}{d\tau} \quad -(7)$$

The laws of conservation of energy and momentum is obeyed throughout the development of the theory. This means that the Newton and Coulomb laws are obtained from spacetime while conserving H, E and L.

With these definitions, eq. (5) is:

$$H = \frac{1}{2} mc^2 = \frac{1}{2} \left(1 - \frac{r_0}{r}\right)^{-1} \frac{E}{mc^2} - \frac{1}{2} \left(1 - \frac{r_0}{r}\right)^{-1} m \left(\frac{dr}{d\tau}\right)^2 - \frac{1}{2} \frac{L^2}{mr^2} \quad -(8)$$

In the limit: $r \to \infty$ $\quad -(9)$

eq. (8) simplifies to:

3)

$$\therefore H = \frac{1}{2}mc^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - \left(m\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{mr^2}\right)\right) \quad -(10)$$

Corresponding to the metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\underline{r}\cdot d\underline{r} \quad -(11)$$

where

$$d\underline{r}\cdot d\underline{r} = dr^2 + r^2 d\phi^2 \quad -(12)$$

$$= v^2 dt^2$$

by definition.

From eq. (11):

$$\gamma^2 d\tau^2 = dt^2 \quad -(13)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(14)$$

Therefore

$$mv^2 = m\left(\frac{dr}{dt}\right)^2 + mr^2\left(\frac{d\phi}{dt}\right)^2 \quad -(15)$$

and

$$\gamma^2 mv^2 = m\left(\frac{dr}{d\tau}\right)^2 + mr^2\left(\frac{d\phi}{d\tau}\right)^2 \quad -(16)$$

The <u>relativistic momentum</u> is:

$$p = \gamma m v \quad -(17)$$

so

$$\frac{p^2}{m} = \gamma^2 mv^2 \quad -(18)$$

and

$$\boxed{H = \frac{1}{2}mc^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - \frac{p^2}{m}\right)} \quad -(19)$$

4) which is the Einstein energy equation:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(20)$$

where:

$$\frac{p^2}{2m} = \frac{1}{2} m \left(\left(\frac{dr}{d\tau} \right)^2 + r^2 \left(\frac{d\phi}{d\tau} \right)^2 \right)$$

$$\boxed{\frac{p^2}{2m} = \frac{1}{2} m \left(\frac{dr}{d\tau} \right)^2 + \frac{1}{2} \frac{L^2}{m r^2}} \quad -(21)$$

Eq. (20) is:

$$\boxed{H^2 = \frac{m^2 c^4}{4} = \frac{1}{4} \left(E^2 - c^2 p^2 \right)} \quad -(22)$$

Eq. (22) implies that the relativistic momentum p is also conserved, because H and E are conserved.

From eqs. (20):

$$\frac{p^2}{2m} = \frac{1}{2mc^2} \left(E^2 - m^2 c^4 \right) \quad -(23)$$

$$\boxed{\frac{p^2}{2m} = \frac{E^2}{2mc^2} - H} \quad -(24)$$

Eq. (20) is: $p^\mu p_\mu = m^2 c^4$ $\quad -(25)$

5) where $p^\mu = \left(\frac{E}{c}, \underline{p}\right)$ — (26)

$$p_\mu = \left(\frac{E}{c}, -\underline{p}\right) \quad -(27)$$

so: $\boxed{H = \frac{1}{2m} p^\mu p_\mu = \frac{1}{2} mc^2}$ — (28)

This is the usual format of the Hamiltonian.

In the presence of the four potential of the electromagnetic field:

$$A^\mu = \left(\phi, c\underline{A}\right) \quad -(29)$$

$$H = \frac{1}{2m}\left(p^\mu - eA^\mu\right)\left(p_\mu - eA_\mu\right) = \frac{1}{2} mc^2 \quad -(30)$$

and this is the relativistic Hamilton Jacobi equation.

Using $p^\mu = i\hbar \partial^\mu$ — (31)

eq. (28) gives the Klein Gordon equation.

$$\left(\square + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \quad -(32)$$

and the Dirac equation:

$$\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right)\psi = 0 \quad -(33)$$

All these equations emerge from spacetime, and all conserve H, E, p, and L.