1) **151(5): Reduction of the General Metric to the Format of a Minkowski Metric.**

Start by considering the Newtonian metric:

$$ds^2 = dr^2 + r^2 d\phi^2 \quad -(1)$$

in the XY plane. The velocity is:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 \quad -(2)$$

and the metric does not give any relation between dr and $d\phi$. Also, dt does not appear in the metric.

The Minkowski metric is:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(3)$$

in which

$$d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\phi^2 \quad -(4)$$

and

$$\frac{d\phi}{dr} = \frac{1}{r^2}\left(\frac{1}{b^2} - \frac{1}{a^2} - \frac{1}{r^2}\right)^{-1/2}$$

$$:= f_1 \quad -(5)$$

Therefore:

$$\boxed{d\underline{r} \cdot d\underline{r} = \left(1 + r^2 f_1^2\right) dr^2} \quad -(6)$$

By definition:

$$v^2 = d\underline{r} \cdot d\underline{r} \,/\, dt^2 \quad -(7)$$

so

$$\boxed{\begin{aligned} v^2 &= \left(1 + r^2 f_1^2\right)\left(\frac{dr}{dt}\right)^2 \\ d\tau^2 &= \left(1 - \frac{v^2}{c^2}\right) dt^2 \end{aligned}} \quad -(8)$$

2) The gravitational metric is:

$$ds^2 = c^2 d\tau^2 = c^2\left(1-\frac{r_0}{r}\right)dt^2 - \left(1-\frac{r_0}{r}\right)^{-1}dr^2 - r^2 d\phi^2 \qquad -(9)$$

Write this as:

$$ds'^2 = c^2 d\tau'^2 = c^2 dt'^2 - d\underline{r}'\cdot d\underline{r}' \qquad -(10)$$

in the format of a Minkowski metric.

By definition:

$$d\underline{r}'\cdot d\underline{r}' = \left(1-\frac{r_0}{r}\right)^{-1}dr^2 - r^2 d\phi^2 \qquad -(11)$$

$$dt'^2 = \left(1-\frac{r_0}{r}\right)dt^2 \qquad -(12)$$

$$v'^2 = d\underline{r}'\cdot d\underline{r}' / dt'^2 \qquad -(13)$$

$$d\tau'^2 = \left(1-\frac{v'^2}{c^2}\right)dt'^2 \qquad -(14)$$

From eq. (9):

$$\frac{d\phi}{dr} = \frac{1}{r^2}\left(\frac{1}{b^2} - \left(1-\frac{r_0}{r}\right)\left(\frac{1}{a^2}+\frac{1}{r^2}\right)\right)^{-1/2} = f_2 \qquad -(15)$$

so

$$d\underline{r}'\cdot d\underline{r}' = \left(\left(1-\frac{r_0}{r}\right)^{-1} + r^2 f_2\right)dr^2$$

$$= \alpha\, d\underline{r}\cdot d\underline{r} \qquad -(16)$$

where

$$\boxed{\alpha = \frac{\left(1-\frac{r_0}{r}\right)^{-1} + r^2 f_2}{1 + r^2 f_1}} \qquad -(17)$$

3)

Therefore:

$$\boxed{d\underline{r}' \cdot d\underline{r}' = \alpha \, d\underline{r} \cdot d\underline{r}} \quad -(18)$$

As

$$r \to \infty \quad -(19)$$

then

$$\alpha \to 1 \quad -(20)$$

From these results:

$$\underline{v}' = \alpha \frac{d\underline{r} \cdot d\underline{r}}{dt'^2} \quad -(21)$$

i.e.

$$\boxed{\underline{v}' = \left(1 - \frac{r_0}{r}\right)^{-1} \alpha \, \underline{v}} \quad -(22)$$

Finally

$$d\tau'^2 = \left(1 - \frac{v'^2}{c^2}\right) dt'^2 \quad -(23)$$

$$\boxed{d\tau'^2 = \left(1 - \frac{v'^2}{c^2}\right)\left(1 - \frac{r_0}{r}\right) dt^2} \quad -(24)$$

Similarly, the general metric for the Orbital Theorem:

$$ds^2 = c^2 dt^2 = x c^2 dt^2 - \frac{1}{x} dr^2 - d\phi^2 \quad -(25)$$

can be reduced to the form of a Minkowski metric.

This is <u>the Principle of Orbits.</u>