notes 151(4): Light Deflection for the General Orbit.

The general orbit in the XY plane is defined by:

$$\phi = f(r) \quad -(1)$$

so the light deflection due to the general orbit is:

$$\Delta\phi = 2\int_{R_0}^{\infty} \left(\frac{d\phi}{dr}\right) dr \quad -(2)$$

where

$$\frac{d\phi}{dr} = f'(r) \quad -(3)$$

The general metric is the Minkowski metric constrained by eq. (3), i.e.:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 \quad -(4)$$
$$= c^2 dt^2 - \left(\frac{1}{f'^2} + r^2\right) d\phi^2$$

The general lagrangian is:

$$L = T = \frac{1}{2}mc^2 = \frac{1}{2}mc^2\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{2}m\left(\frac{1}{f'^2} + r^2\right)\left(\frac{d\phi}{d\tau}\right)^2$$
$$-(5)$$

The constants of motion are the total energy:

$$E = mc^2\left(\frac{dt}{d\tau}\right)^2 = \gamma^2 mc^2 \quad -(6)$$

the total angular momentum:

$$L = m\left(r^2 + \frac{1}{f'^2}\right)\left(\frac{d\phi}{d\tau}\right)^2 \quad -(7)$$

where

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} = \gamma\omega \quad -(8)$$

where

$$\omega = \frac{d\phi}{dt} \quad -(9)$$

is the orbital angular velocity. So

$$\boxed{L = m\left(r^2 + \frac{1}{f'^2}\right)\gamma^2\omega^2} \quad -(10)$$

for any orbit in XY.

The equation of motion of any orbit in XY is:

$$\boxed{\frac{E^2}{mc^2} - mc^2 = L^2} \quad -(11)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(12)$$

In the limit:

$$v \ll c \quad -(13)$$

$$\frac{E^2}{mc^2} - mc^2 = mc^2\left(\gamma^2 - 1\right)$$

$$= mc^2\left(\left(1 - \frac{v^2}{c^2}\right)^{-1} - 1\right)$$

$$\rightarrow mc^2\frac{v^2}{c^2} = mv^2 \quad -(14)$$

3)

The right hand side of eq. (11) becomes

$$L \to \gamma\, m\omega^2\left(r^2 + \frac{1}{f'^2}\right) \quad -(15)$$

as

$$\gamma^2 \to 1 \quad -(16)$$

So in the limit (13), eq. (10) becomes:

$$\boxed{v^2 = \omega^2\left(r^2 + \frac{1}{f'^2}\right)} \quad -(17)$$

This is:

$$v^2 = \omega^2\left(r^2 + \left(\frac{dr}{d\phi}\right)^2\right) \quad -(18)$$

For a circular orbit:

$$dr = 0 \quad -(19)$$

and

$$v = \omega r. \quad -(20)$$

In general:

$$\boxed{\frac{E^2}{mc^2} - mc^2 = m\gamma^2\omega^2\left(r^2 + \left(\frac{dr}{d\phi}\right)^2\right)} \quad -(21)$$

which is the equation of motion of all orbits in XY.

The mathematical description is:

$$m\gamma^2\omega^2\left(r^2+\left(\frac{dr}{d\phi}\right)^2\right) = x \quad -(22)$$

where

$$x = \text{constant} = \frac{E^2}{mc^2} - mc^2 \quad -(23)$$

In a formal sense, $\frac{dr}{d\phi}$ and $\frac{d\phi}{dr}$ can be found from observation of x, γ, ω, ~~and~~ and m.

Newtonian Orbits

In this case:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(24)$$

where α and ϵ are found by observation of the orbit. Therefore:

$$\frac{dr}{d\phi} = \frac{\epsilon\sin\phi}{(1+\epsilon\cos\phi)^2} = \left(\frac{\epsilon r^2}{\alpha^2}\right)\sin\phi \quad -(25)$$

In Newtonian orbits, $v \ll c$, so eq. (18) can be ed, and:

$$\boxed{v^2 = \omega^2 r^2\left(1+\left(\frac{r^2\epsilon\sin\phi}{\alpha}\right)^2\right)} \quad -(26)$$

As

$$\epsilon \rightarrow 0 \quad -(27)$$

a circular orbit is obtained:

$$v = \omega r \quad -(28)$$

5)

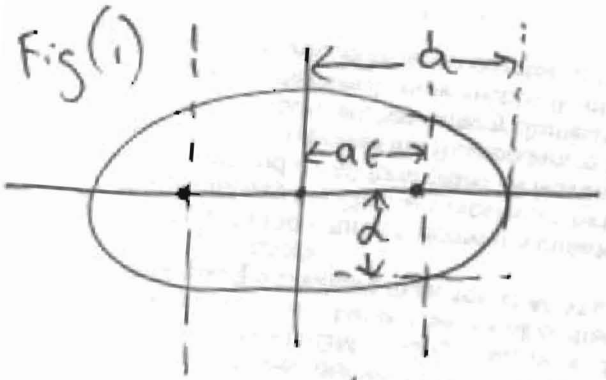
Eq. (26) appears to be a new law of Newtonian orbits. Here α is the latus rectum:

$$\alpha = \frac{L^2}{\mu k} \quad -(29)$$

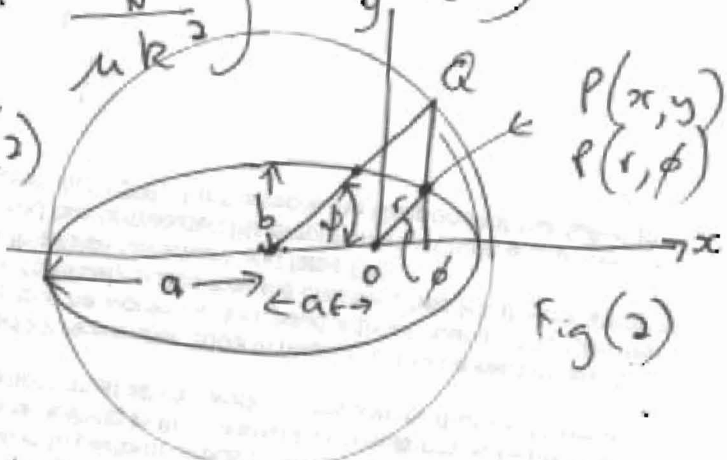
and ϵ is the eccentricity.

$$\epsilon = \left(1 + \frac{2E_N L^2}{\mu k^2}\right)^{1/2} \quad -(30)$$

Fig(1)



Fig(2)



Fig(2)

In eqs. (29) and (30)

$$\mu = \frac{mM}{m+M} \quad -(31)$$

$$L = \mu r^2 \omega, \quad k = mMG \quad -(32)$$

$$E_N = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{L^2}{\mu r^2} - \frac{k}{r} \quad -(33)$$

$$= \frac{1}{2} m v^2 - \frac{k}{r}$$

From Kepler's construction in Fig (2) the eccentric anomaly is:

$$\cos\psi = (x + a\epsilon)/a \quad -(34)$$

$$\sin\psi = y/b \quad -(35)$$

and the mean anomaly is

$$M = \psi - \epsilon\sin\psi \quad -(36)$$

$$= 2\pi t/\tau$$

) where τ is the time for one orbit. Therefore

$$v^2 = \frac{k}{\mu}\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(37)$$

which is Kepler's equation

Comparing eqs. (26) and (37):

$$\omega^2 r^2\left(1 + \left(\frac{r\epsilon\sin\phi}{\alpha}\right)^2\right) = \frac{k}{\mu}\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(38)$$

For a circular orbit:

$$\omega^2 r^3 = \frac{k}{\mu} = \frac{mMG(m+M)}{mM}$$

$$\boxed{\omega^2 r^3 = (m+M)G} \quad -(39)$$

2) Relativistic Keplerian Orbits

In this case:

$$r = \frac{\alpha}{1+\epsilon\cos(y\phi)} \quad -(40)$$

and

$$\frac{E^2}{mc^2} - mc^2 = m\gamma^2\omega^2\left(r^2 + \left(\frac{dr}{d\phi}\right)^2\right) \quad -(41)$$

where

$$\frac{dr}{d\phi} = \left(\frac{y\epsilon r^2}{\alpha^2}\right)\sin\phi \quad -(42)$$

For any orbit in XY:

$$v^2 dt^2 = c^2(dt^2 - d\tau^2) = \left(\frac{1}{f'^2} + r^2\right)d\phi^2 \quad -(43)$$

where $$v^2 = d\underline{r} \cdot d\underline{r} \,/\, dt^2 \quad -(44)$$

and $$d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\phi^2$$
$$= \left(\frac{1}{f'^2} + r^2 \right) d\phi^2 \quad -(45)$$

i.e. $$\boxed{v^2 = \left(\frac{1}{f'^2} + r^2 \right) \left(\frac{d\phi}{dt} \right)^2} \quad -(46)$$

This is a fully relativistic result, and equivalent to eq. (11).

Using eq. (3):

$$v^2 = \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\phi}{dt} \right)^2 \right) \quad -(47)$$

The simplest form of the equation of orbits is therefore

$$\boxed{\left(\frac{dr}{d\phi} \right)^2 = \left(\frac{v}{\omega} \right)^2 - r^2} \quad -(48)$$

and this is valid for all orbits.